ANTENNAS AND WAVE PROPAGATION

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CHAPTER 1

Electromagnetic Radiation

Introduction

Most of us are familiar with cellular phones. In cellular communication systems, there is a two-way wireless transmission between the cellular phone handset and the base station tower. The cell phone converts the audio signals into electrical form using a microphone. This information is imposed on a high frequency carrier signal by the process of modulation. The modulated carrier is radiated into free space as an electromagnetic wave which is picked up by the base station tower. Similarly, the signals transmitted by the tower are received by the handset, thus establishing a two way communication. This is one of the typical examples of a wireless communication system which uses free space as a medium to transfer information from the transmitter to the receiver. A key component of a wireless link is the antenna which efficiently couples electromagnetic energy from the transmitter to free space and from free space to the receiver. An antenna is generally a bidirectional device, i.e., the power through the antenna can flow in both the directions, hence it works as a transmitting as well as a receiving antenna.

Transmission lines are used to transfer electromagnetic energy from one point to another within a circuit and this mode of energy transfer is generally known as *guided wave* propagation. An antenna acts as an interface between the radiated electromagnetic waves and the guided waves. It can be thought of as a mode transformer which transforms a guided-wave field distribution into a radiated-wave field distribution. Since the wave impedances of the guided and the radiated waves may be different, the antenna can also be thought of as an impedance transformer. A proper design of this part is necessary for the efficient coupling of the energy from the circuit to the free space and vice versa.

One of the important properties of an antenna is its ability to transmit power in a preferred direction. The angular distribution of the transmitted

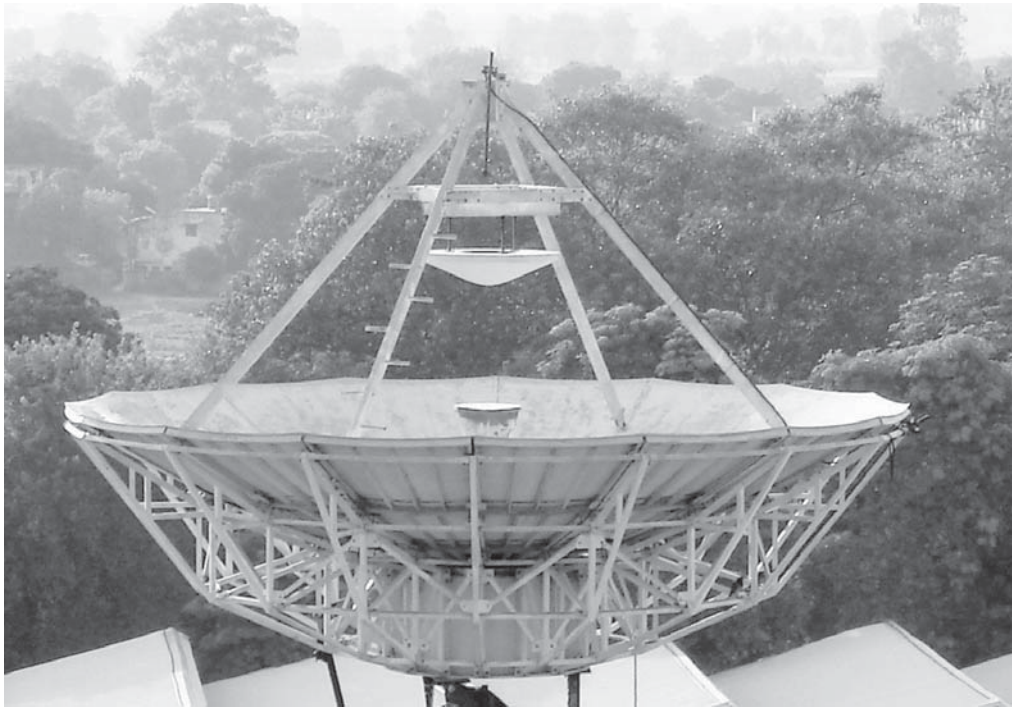

Fig. 1.1 Parabolic dish antenna at the Department of Electrical Engineering, Indian Institute of Technology, Kanpur, India (Courtesy: Dept of EE, IIT Kanpur)

power around the antenna is generally known as the *radiation pattern* (A more precise definition is given in Chapter 2). For example, a cellular phone needs to communicate with a tower which could be in any direction, hence the cellular phone antenna needs to radiate equally in all directions. Similarly, the tower antenna also needs to communicate with cellular phones located all around it, hence its radiation also needs to be independent of the direction.

There are large varieties of communication applications where the directional property is used to an advantage. For example, in point-to-point communication between two towers it is sufficient to radiate (or receive) only in the direction of the other tower. In such cases a highly directional parabolic dish antenna can be used. A 6.3 m diameter parabolic dish antenna used for communication with a geo-stationary satellite is shown in Fig. 1.1. This antenna radiates energy in a very narrow beam pointing towards the satellite.

Radio astronomy is another area where highly directional antennas are used. In radio astronomy the antenna is used for receiving the electromagnetic radiations from outer space. The power density of these signals from outer space is very low, hence it is necessary to collect the energy over a very large area for it to be useful for scientific studies. Therefore, radio astronomy antennas are large in size. In order to increase the collecting aperture,

Fig. 1.2 A panoramic view of the Giant Metrewave Radio Telescope (GMRT), Pune, India, consisting of 30 fully-steerable parabolic dish antennas of 45 m diameter each spread over distances up to 25 km.[1] (Photograph by Mr. Pravin Raybole, Courtesy: GMRT, Pune, http://www.gmrt.ncra.tifr.res.in)

the Giant Metrewave Radio Telescope (GMRT) near Pune in India, has an array of large dish antennas, as shown in Fig. 1.2.

The ability of an antenna to concentrate power in a narrow beam depends on the size of the antenna in terms of wavelength. Electromagnetic waves of wavelengths ranging from a few millimetres to several kilometres are used in various applications requiring efficient antennas working at these wavelengths. These frequencies, ranging from hundreds of giga hertz to a few kilo hertz, form the radio wave spectrum. Figure 1.3 depicts the radio wave spectrum along with band designations and typical applications.

The radiation pattern of an antenna is usually computed assuming the surroundings to be infinite free space in which the power density (power per unit area) decays as inverse square of the distance from the antenna. In practical situations the environment is more complex and the decay is not as simple. If the environment consists of well defined, finite number of scatterers, we can use theories of reflection, refraction, diffraction, etc., to predict the propagation of electromagnetic waves. However, in a complex environment, such as a cell phone operating in an urban area, the field strength is obtained by empirical relations.

The atmosphere plays a significant role in the propagation of electromagnetic waves. The density of the air molecules and, hence, the refractive index of the atmosphere changes with height. An electromagnetic wave passing through media having different refractive indices undergoes refraction. Thus, the path traced by an electromagnetic wave as it propagates through

[1]The GMRT was built and is operated by the National Centre for Radio Astrophysics (NCRA) of the Tata Institute of Fundamental Research (TIFR).

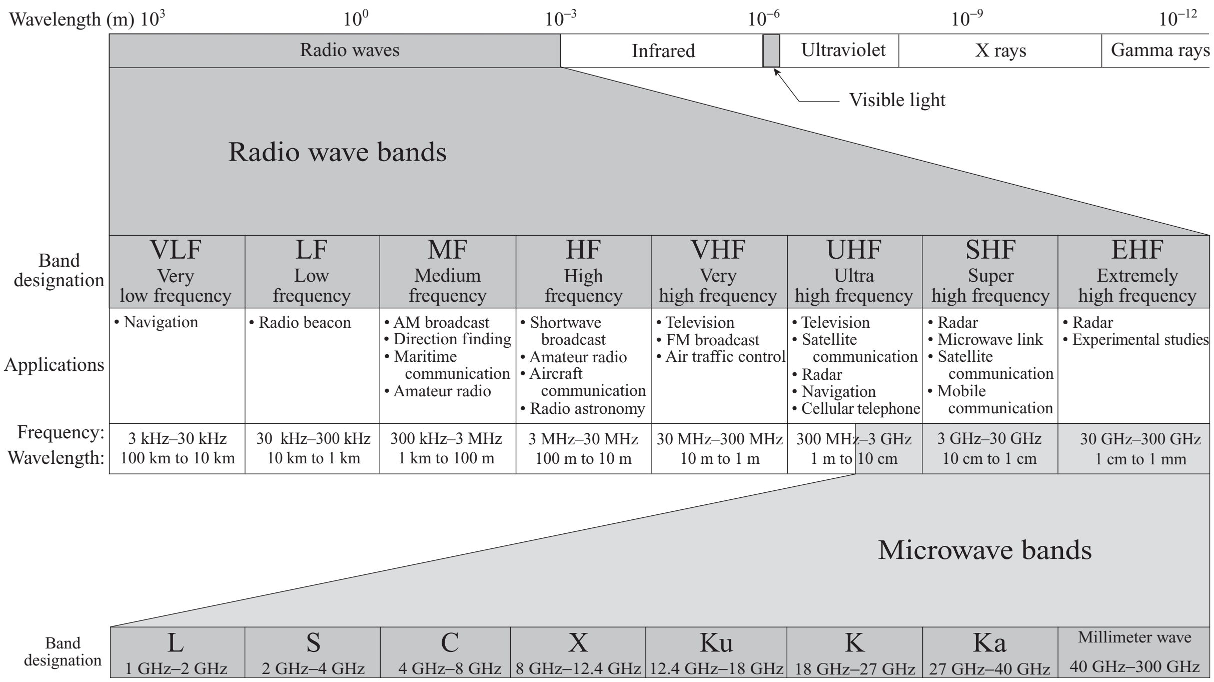

Fig. 1.3 Radio wave spectrum along with the band designations and typical applications.

the atmosphere is not a straight line. The air molecules also get ionized by solar radiation and cosmic rays. The layer of ionized particles in the atmosphere, known as the *ionosphere*, reflects high frequency (3 MHz to 30 MHz) waves. A multi-hop communication link is established by repeated reflections of the electromagnetic waves between the ionosphere and the surface of the earth. This is the mode of propagation of shortwave radio signals over several thousand kilometres.

Both the radiation properties of the antennas and the propagation conditions play a very important role in establishing a successful communication link. This book addresses both these issues in some detail. It is assumed that the students have some basic knowledge of electromagnetic theory. However, in the following section some of the basic concepts of electromagnetic theory used in the analysis of antennas are presented for easy reference as well as for introducing the notation used in the book.

1.1 Review of Electromagnetic Theory

Electromagnetic fields are produced by time-varying charge distributions which can be supported by time-varying current distributions. Consider sinusoidally varying electromagnetic sources. (Sources having arbitrary variation with respect to time can be represented in terms of sinusoidally varying functions using Fourier analysis.) A sinusoidally varying current $i(t)$ can be expressed as a function of time, t, as

$$i(t) = I_0 \cos(\omega t + \varphi) \tag{1.1}$$

where I_0 is the amplitude (unit: ampere, A), ω is the angular frequency (unit: radian per second, rad/s), and φ is the phase (unit: radian, rad). The angular frequency, ω, is related to the frequency, f (unit: cycle per second or Hz), by the relation $\omega = 2\pi f$. One may also express the current $i(t)$ as a sine function

$$i(t) = I_0 \sin(\omega t + \varphi') \tag{1.2}$$

where $\varphi' = \varphi + \pi/2$. Therefore, we need to identify whether the phase has been defined taking the cosine function or the sine function as a reference. In this text, we have chosen the cosine function as the reference to define the phase of the sinusoidal quantity.

Since $\cos(\omega t + \varphi) = \mathrm{Re}\left\{e^{j(\omega t+\varphi)}\right\}$ where, $\mathrm{Re}\{\}$ represents the real part of the quantity within the curly brackets, the current can now be written as

$$i(t) = I_0 \mathrm{Re}\left\{e^{j(\omega t+\varphi)}\right\} \tag{1.3}$$

$$= \mathrm{Re}\left\{I_0 e^{j\varphi} e^{j\omega t}\right\} \tag{1.4}$$

The quantity $I_0 e^{j\varphi}$ is known as a phasor and contains the amplitude and phase information of $i(t)$ but is independent of time, t.

EXAMPLE 1.1

Express $i(t) = (\cos\omega t + 2\sin\omega t)$ A in phasor form.

Solution: First we must express $\sin\omega t$ in terms of the cosine function using the relation $\cos(\omega t - \pi/2) = \sin\omega t$. Therefore

$$i(t) = \cos\omega t + 2\cos\left(\omega t - \frac{\pi}{2}\right)$$

Using the relation $\cos(\omega t + \varphi) = \mathrm{Re}\left\{e^{j(\omega t+\varphi)}\right\}$

$$i(t) = \mathrm{Re}\left\{e^{j\omega t}\right\} + \mathrm{Re}\left\{2e^{j(\omega t-\pi/2)}\right\}$$

For any two complex quantities Z_1 and Z_2, $\mathrm{Re}\{Z_1 + Z_2\} = \mathrm{Re}\{Z_1\} + \mathrm{Re}\{Z_2\}$ and, hence, the current can be written as

$$\begin{aligned} i(t) &= \mathrm{Re}\{(1 + 2e^{-j\pi/2})e^{j\omega t}\} \\ &= \mathrm{Re}\{(1 - j2)e^{j\omega t}\} \\ &= \mathrm{Re}\{2.24e^{-j1.1071}e^{j\omega t}\} \end{aligned}$$

Therefore, in the phasor notation the current is given by

$$I = 2.24e^{-j1.1071} \text{ A}$$

EXAMPLE 1.2

Express the phasor current $I = (I_1 e^{j\varphi_1} + I_2 e^{j\varphi_2})$ as a function of time.

Solution: The instantaneous current can be expressed as

$$i(t) = \mathrm{Re}\{I e^{j\omega t}\}$$

Substituting the value of I

$$\begin{aligned} i(t) &= \mathrm{Re}\{I_1 e^{j\varphi_1} e^{j\omega t} + I_2 e^{j\varphi_2} e^{j\omega t}\} \\ &= I_1 \cos(\omega t + \varphi_1) + I_2 \cos(\omega t + \varphi_2) \end{aligned}$$

The field vectors that vary with space, and are sinusoidal functions of time, can also be represented by phasors. For example, an electric field vector $\bar{\mathcal{E}}(x, y, z, t)$, a function of space (x, y, z) having a sinusoidal variation with time, can be written as

$$\bar{\mathcal{E}}(x, y, z, t) = \mathrm{Re}\Big\{\mathbf{E}(x, y, z)e^{j\omega t}\Big\} \tag{1.5}$$

where $\mathbf{E}(x, y, z)$ is a phasor that contains the direction, magnitude, and phase information of the electric field, but is independent of time. In the text that follows, $e^{j\omega t}$ time variation is implied in all the field and source quantities and is not written explicitly. In this text, bold face symbols (e.g., $\mathbf{E}$) are used for vectors, phasors, or matrices, italic characters for scalar quantities (e.g., t), script characters (e.g., $\mathcal{E}$) for instantaneous scalar quantities, and script characters with an over-bar (e.g., $\bar{\mathcal{E}}$) for instantaneous vector quantities.

Using phasor notation, Maxwell's equations can be written for the fields and sources that are sinusoidally varying with time as[1]

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{1.6}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \mathbf{J} \tag{1.7}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{1.8}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.9}$$

The symbols used in Eqns (1.6) to (1.9) are explained below:

E : Electric field intensity (unit: volt per metre, V/m)
H : Magnetic field intensity (unit: ampere per metre, A/m)
D : Electric flux density (unit: coulomb per metre, C/m)
B : Magnetic flux density (unit: weber per metre, Wb/m or tesla, T)
J : Current density (unit: ampere per square metre, A/m^2)
ρ : Charge density (unit: coulomb per cubic metre, C/m^3)

The first two curl equations are the mathematical representations of Faraday's and Ampere's laws, respectively. The divergence equation [Eqn (1.8)] represents Gauss's law. Since magnetic monopoles do not exist in nature, we have zero divergence for **B** [Eqn (1.9)].

The current density, **J**, consists of two components. One is due to the impressed or actual sources and the other is the current induced due to the applied electric field, which is equal to $\sigma\mathbf{E}$, where σ is the conductivity of the medium (unit: siemens per metre, S/m). In antenna problems, we are mostly working with fields radiated into free space with $\sigma = 0$. Therefore, in the analyses that follow, unless explicitly specified, **J** represents impressed-source current density.

In an isotropic and homogeneous medium, the electric flux density, **D**, and the electric field intensity, **E**, are related by

$$\mathbf{D} = \epsilon\mathbf{E} \tag{1.10}$$

where ϵ is the electric permittivity (unit: farad per metre, F/m) of the medium. ϵ_0 is the permittivity of free space ($\epsilon_0 = 8.854 \times 10^{-12}$ F/m) and the ratio, $\epsilon/\epsilon_0 = \epsilon_r$ is known as the relative permittivity of the medium. It is

[1]See Cheng 2002, Hayt et al. 2001, Jordan et al. 2004, and Ramo et al. 2004.

a dimensionless quantity. Similarly, magnetic flux density, $\mathbf{B}$, and magnetic field intensity, $\mathbf{H}$, are related by

$$\mathbf{B} = \mu \mathbf{H} \tag{1.11}$$

where $\mu = \mu_0 \mu_r$ is the magnetic permeability (unit: henry per metre, H/m) of the medium. μ_0 is the permeability of free space ($\mu_0 = 4\pi \times 10^{-7}$ H/m) and the ratio, $\mu/\mu_0 = \mu_r$, is known as the relative permeability of the medium. For an isotropic medium ϵ and μ are scalars and for a homogeneous medium they are independent of position.

One of the problems in antenna analysis is that of finding the $\mathbf{E}$ and $\mathbf{H}$ fields in the space surrounding the antenna. An antenna operating in the transmit mode is normally excited at a particular input point in the antenna structure. (The same point is connected to the receiver in the receive mode). Given an antenna structure and an input excitation, the current distribution on the antenna structure is established in such a manner that Maxwell's equations are satisfied everywhere and at all times (along with the boundary conditions which, again, are derived from Maxwell's equations using the behaviour of the fields at material boundaries). The antenna analysis can be split into two parts—(a) determination of the current distribution on the structure due to the excitation and (b) evaluation of the field due to this current distribution in the space surrounding the antenna. The first part generally leads to an integral equation, the treatment of which is beyond the scope of this book. We will be mainly concerned with the second part, i.e., establishing the antenna fields, given the current distribution.

Maxwell's equations [Eqns (1.6)–(1.9)] are time-independent, first order differential equations to be solved simultaneously. It is a common practice to reduce these equations to two second order differential equations called wave equations. For example, in a source-free region ($\rho = 0$ and $\mathbf{J} = 0$) we can take the curl of the first equation [Eqn (1.6)], substitute it in the second equation [Eqn (1.7)] to eliminate $\mathbf{H}$, and get the wave equation, $\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0$, satisfied by the $\mathbf{E}$ field. Similarly, we can also derive the wave equation satisfied by the $\mathbf{H}$ field. (Start from the curl of Eqn (1.7) and substitute in Eqn (1.6) to eliminate $\mathbf{E}$). Thus, it is sufficient to solve one equation to find both $\mathbf{E}$ and $\mathbf{H}$ fields, since they satisfy the same wave equation.

To simplify the problem of finding the $\mathbf{E}$ and $\mathbf{H}$ fields due to a current distribution, we can split it into two parts by defining intermediate potential functions which are related to the $\mathbf{E}$ and $\mathbf{H}$ fields. This is known as the vector potential approach and is discussed in the following subsection.

1.1.1 Vector Potential Approach

Given a current distribution on the antenna, the problem is one of determining the **E** and **H** fields due to this current distribution which satisfies all four of Maxwell's equations along with the boundary conditions, if any. In the vector potential approach we carry out the solution to this problem in two steps by defining intermediate potential functions. In the first step, we determine the potential function due to the current distribution and in the second step, the **E** and **H** fields are computed from the potential function. In the analysis that follows, the relationships between the vector potential and the current distribution as well as the **E** and **H** fields are derived. All four of Maxwell's equations are embedded in these relationships.

Let us start with the last of the Maxwell's equations, $\nabla \cdot \mathbf{B} = 0$. Since the curl of a vector field is divergence-free (vector identity: $\nabla \cdot \nabla \times \mathbf{A} = 0$), **B** can be expressed as a curl of an arbitrary vector field, **A**. We call this a magnetic vector potential function because it is related to the magnetic flux density, **B**, via the relationship

$$\mu \mathbf{H} = \mathbf{B} = \nabla \times \mathbf{A} \tag{1.12}$$

or

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \tag{1.13}$$

Substituting this into the equation $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$, Maxwell's first equation is also incorporated

$$\nabla \times \mathbf{E} = -j\omega(\mu\mathbf{H}) = -j\omega(\nabla \times \mathbf{A}) \tag{1.14}$$

or

$$\nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0 \tag{1.15}$$

Since the curl of a gradient function is zero (vector identity: $\nabla \times \nabla V = 0$), the above equation suggests that the quantity in brackets can be replaced by the gradient of a scalar function. Specifically, a scalar potential function V is defined such that

$$(\mathbf{E} + j\omega\mathbf{A}) = -\nabla V \tag{1.16}$$

Using this we relate the **E** field to the potential functions as

$$\mathbf{E} = -(\nabla V + j\omega\mathbf{A}) \tag{1.17}$$

Equations (1.13) and (1.17) relate the **H** and **E** fields to the potential functions **A** and V. Now, to satisfy Maxwell's second equation, $\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \mathbf{J}$, substitute the expression for the **E** and **H** fields in terms of the potential functions [Eqns (1.13) and (1.17)]

$$\frac{1}{\mu}\nabla \times (\nabla \times \mathbf{A}) = -j\omega\epsilon(\nabla V + j\omega\mathbf{A}) + \mathbf{J} \tag{1.18}$$

which is valid for a homogeneous medium. Expanding the left hand side using the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{1.19}$$

we have

$$\nabla^2\mathbf{A} + \omega^2\mu\epsilon\mathbf{A} = -\mu\mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + j\omega\mu\epsilon V) \tag{1.20}$$

So far we have satisfied three of Maxwell's four equations. Note that only the curl of **A** is defined so far. Since the curl and divergence are two independent parts of any vector field, we can now define the divergence of **A**. We define $\nabla \cdot \mathbf{A}$ so as to relate **A** and V as well as simplify Eqn (1.20) by eliminating the second term on the right hand side of the equation. We relate **A** and V by the equation

$$\nabla \cdot \mathbf{A} = -j\omega\mu\epsilon V \tag{1.21}$$

This relationship is known as the Lorentz condition. With this the magnetic vector potential, **A**, satisfies the vector wave equation

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J} \tag{1.22}$$

where

$$k = \omega\sqrt{\mu\epsilon} \tag{1.23}$$

is the propagation constant (unit: radian per metre, rad/m) in the medium.

Now, to satisfy Maxwell's fourth equation, $\nabla \cdot \mathbf{D} = \rho$, we substitute $\mathbf{E} = -(\nabla V + j\omega\mathbf{A})$ in this equation to get

$$\nabla \cdot (-\nabla V - j\omega\mathbf{A}) = \frac{\rho}{\epsilon} \tag{1.24}$$

or

$$\nabla^2 V + j\omega(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon} \tag{1.25}$$

Eliminating $\mathbf{A}$ from this equation using the Lorentz condition [Eqn (1.21)]

$$\nabla^2 V + k^2 V = -\frac{\rho}{\epsilon} \tag{1.26}$$

Thus, both $\mathbf{A}$ and V must satisfy the wave equation, the source function being the current density for the magnetic vector potential, $\mathbf{A}$, and the charge density for the electric scalar potential function V.

1.1.2 Solution of the Wave Equation

Consider a spherically symmetric charge distribution of finite volume, V', centred on the origin. Our goal is to compute the scalar potential $V(x, y, z)$ [or $V(r, \theta, \phi)$[1]] due to this source, which is the solution of the inhomogeneous wave equation as given by Eqn (1.26). Since the charge is spherically symmetric, we will solve the wave equation in the spherical coordinate system. The Laplacian $\nabla^2 V$ in the spherical coordinate system[2] is written as

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \tag{1.27}$$

The scalar potential, $V(r, \theta, \phi)$, produced by a spherically symmetric charge distribution is independent of θ and ϕ. Therefore, the wave equation, Eqn (1.26), reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + k^2 V = -\frac{\rho}{\epsilon} \tag{1.28}$$

The right hand side of this equation is zero everywhere except at the source itself. Therefore, in the source-free region, V satisfies the homogeneous wave equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + k^2 V = 0 \tag{1.29}$$

The solutions for V are the scalar spherical waves given by

$$V(r) = V_0^{\pm}\frac{e^{\mp jkr}}{r} \tag{1.30}$$

where V_0^+ is a complex amplitude constant and e^{-jkr}/r is a spherical wave travelling in the $+r$-direction. V_0^- is the complex amplitude of the scalar

[1] (x, y, z): rectangular co-ordinates; (r, θ, ϕ): spherical co-ordinates.
[2] See Appendix E for details on the coordinate systems and vector operations in different coordinate systems.

spherical wave e^{jkr}/r travelling in the $-r$-direction. By substituting this in the wave equation, it can be shown that it satisfies the homogeneous wave equation [Eqn (1.29)].

EXAMPLE 1.3

Show that

$$V(r) = V_0^{\pm}\frac{e^{\mp jkr}}{r}$$

are solutions of

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + k^2V = 0$$

Solution: Let us consider the wave travelling in the positive r-direction

$$V(r) = V_0^{+}\frac{e^{-jkr}}{r}$$

Substituting into the left hand side (LHS) of the given equation

$$\begin{aligned}
\text{LHS} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(V_0^{+}\frac{e^{-jkr}}{r}\right)\right] + k^2V_0^{+}\frac{e^{-jkr}}{r} \\
&= V_0^{+}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-\frac{e^{-jkr}}{r^2} - jk\frac{e^{-jkr}}{r}\right)\right] + k^2V_0^{+}\frac{e^{-jkr}}{r} \\
&= V_0^{+}\frac{1}{r^2}\frac{\partial}{\partial r}\left(-e^{-jkr} - jkre^{-jkr}\right) + k^2V_0^{+}\frac{e^{-jkr}}{r} \\
&= V_0^{+}\frac{1}{r^2}\left[jke^{-jkr} - jke^{-jkr} - k^2re^{-jkr}\right] + k^2V_0^{+}\frac{e^{-jkr}}{r} \\
&= 0
\end{aligned}$$

Therefore, the positive wave is a solution of the given differential equation.

Now, let us consider the wave that is travelling along the negative r-direction

$$V(r) = V_0^{-}\frac{e^{jkr}}{r}$$

Substituting into the left hand side of the given differential equation

$$
\begin{aligned}
\text{LHS} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(V_0^-\frac{e^{jkr}}{r}\right)\right] + k^2V_0^-\frac{e^{jkr}}{r} \\
&= V_0^-\frac{1}{r^2}\frac{\partial}{\partial r}\left[-r^2\frac{1}{r^2}e^{jkr} + jkre^{jkr}\right] + k^2V_0^-\frac{e^{jkr}}{r} \\
&= V_0^-\frac{1}{r^2}\left[-jke^{jkr} + jke^{jkr} - rk^2e^{jkr}\right] + k^2V_0^-\frac{e^{jkr}}{r} \\
&= 0
\end{aligned}
$$

The wave travelling in the $-r$-direction satisfies the differential equation, hence it is also a solution.

These are the two solutions of the wave equation in free space and represent spherical waves propagating away from the origin ($+r$-direction) and converging on to the origin ($-r$-direction). Taking physical considerations into account, the wave converging towards the source is discarded.

The instantaneous value of the scalar potential $\mathcal{V}(r,t)$ for the wave propagating in the $+r$-direction can be written as

$$
\mathcal{V}(r,t) = \text{Re}\left\{V_0^+\frac{e^{j(\omega t - kr)}}{r}\right\} \tag{1.31}
$$

Since V_0^+ is a complex quantity, it can be expressed as, $V_0^+ = |V_0^+|e^{j\varphi_v}$, where φ_v is the phase angle of V_0^+. The equation for the constant phase spherical wave front is

$$
\varphi_v + \omega t - kr = \text{const} \tag{1.32}
$$

The velocity of the wave is the rate at which the constant phase front moves with time. Differentiating the expression for the constant phase front surface with respect to time, we get

$$
j\omega - jk\frac{dr}{dt} = 0 \tag{1.33}
$$

This follows from the fact that V_0^+ and, hence, the phase φ_v, is independent of time, i.e., $d\varphi_v/dt = 0$. Therefore, the velocity (v, unit: metre per second, m/s) of the wave can be expressed as

$$v = \frac{dr}{dt} = \frac{\omega}{k} \tag{1.34}$$

Substituting the value of the propagation constant from Eqn (1.23), the wave velocity is

$$v = \frac{\omega}{\omega\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}} \tag{1.35}$$

The velocity of the wave in free space is equal to 3×10^8 m/s. The distance between two points that are separated in phase by 2π radians is known as the wavelength (λ, unit: metre, m) of the wave. Consider two points r_1 and r_2 on the wave with corresponding phases

$$\varphi_1 = \varphi_v + \omega t - kr_1$$
$$\varphi_2 = \varphi_v + \omega t - kr_2$$

such that

$$\varphi_2 - \varphi_1 = k(r_1 - r_2) = k\lambda = 2\pi \tag{1.36}$$

Therefore, the wavelength and the propagation constant are related by

$$k = \frac{2\pi}{\lambda} \tag{1.37}$$

The velocity can be written in terms of the frequency and the wavelength of the wave

$$v = \frac{\omega}{k} = \frac{2\pi f}{2\pi/\lambda} = f\lambda \tag{1.38}$$

EXAMPLE 1.4

The electric field of an electromagnetic wave propagating in a homogeneous medium is given by

$$\bar{\mathcal{E}}(x, y, z, t) = \mathbf{a}_\theta \frac{50}{r} \cos(4\pi \times 10^6 t - 0.063r) \text{ V/m}$$

Calculate the frequency, propagation constant, velocity, and the magnetic field intensity of the wave if the relative permeability of the medium is equal to unity.

Solution: The θ-component of the electric field can be expressed as

$$\mathcal{E}_\theta = \mathrm{Re}\left\{\frac{50}{r}e^{j(4\pi\times10^6 t - 0.063r)}\right\}$$

Comparing this with Eqn (1.31), $\omega = 4\pi \times 10^6$ rad/s, hence frequency of the wave is $f = \omega/(2\pi) = 2$ MHz, and the propagation constant is $k = 0.063$ rad/m. The velocity of the wave is given by $v = \omega/k = 4\pi \times 10^6/0.063 = 2 \times 10^8$ m/s.

Expressing the electric field as a phasor

$$\mathbf{E} = \mathbf{a}_\theta \frac{50}{r} e^{-j0.063r} \text{ V/m}$$

Substituting this in Maxwell's equation, Eqn (1.6), and expressing the curl in spherical coordinates

$$-j\omega\mu\mathbf{H} = \nabla \times \mathbf{E} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta\mathbf{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 0 & rE_\theta & 0 \end{vmatrix}$$

Expanding the determinant

$$\nabla \times \mathbf{E} = \frac{1}{r^2\sin\theta}\left[-\mathbf{a}_r\frac{\partial(rE_\theta)}{\partial\phi} + \mathbf{a}_\phi r\sin\theta\frac{\partial(rE_\theta)}{\partial r}\right]$$

Since r and E_θ are not functions of ϕ

$$\nabla \times \mathbf{E} = \mathbf{a}_\phi \frac{50}{r}(-j0.063)e^{-j0.063}$$

Therefore, the magnetic field is given by

$$\mathbf{H} = \frac{1}{-j\omega\mu}\nabla \times \mathbf{E} = \mathbf{a}_\phi \frac{0.063}{\omega\mu} \times \frac{50}{r}e^{-0.063r}$$

Substituting the values of $\omega = 4\pi \times 10^6$ rad/s and $\mu = 4\pi \times 10^{-7}$ H/m

$$\mathbf{H} = \mathbf{a}_\phi \frac{0.2}{r} e^{-j0.063r} \text{ A/m}$$

The magnetic field can also be expressed as a function of time.

$$\bar{\mathcal{H}} = \mathbf{a}_\phi \frac{0.2}{r} \cos(4\pi \times 10^6 t - 0.063r) \text{ A/m}$$

Consider a static point charge q kept at a point with position vector $\mathbf{r}'$ as shown in Fig. 1.4. The electric potential, V, at a point $P(r, \theta, \phi)$, with the position vector $\mathbf{r}$, is given by

$$V(r, \theta, \phi) = \frac{q}{4\pi\epsilon R} \tag{1.39}$$

where R is the distance between the charge and the observation point, $R = |\mathbf{R}| = |\mathbf{r} - \mathbf{r}'|$ (see Fig. 1.4). We are using two coordinate notations, the primed coordinates (x', y', z') for the source point and the unprimed coordinates (x, y, z) or (r, θ, ϕ) for the field point.

If there are more than one point charges, the potential is obtained by the superposition principle, i.e., summing the contributions of all the point charges. If the source is specified as a charge density distribution over a volume, the potential at any field point is obtained by integration over the source volume. To do this, we first consider a small volume $\Delta v'$ centered on $\mathbf{r}'$. The charge contained in this volume is $\rho(\mathbf{r}')\Delta v'$, where $\rho(\mathbf{r}')$ is the

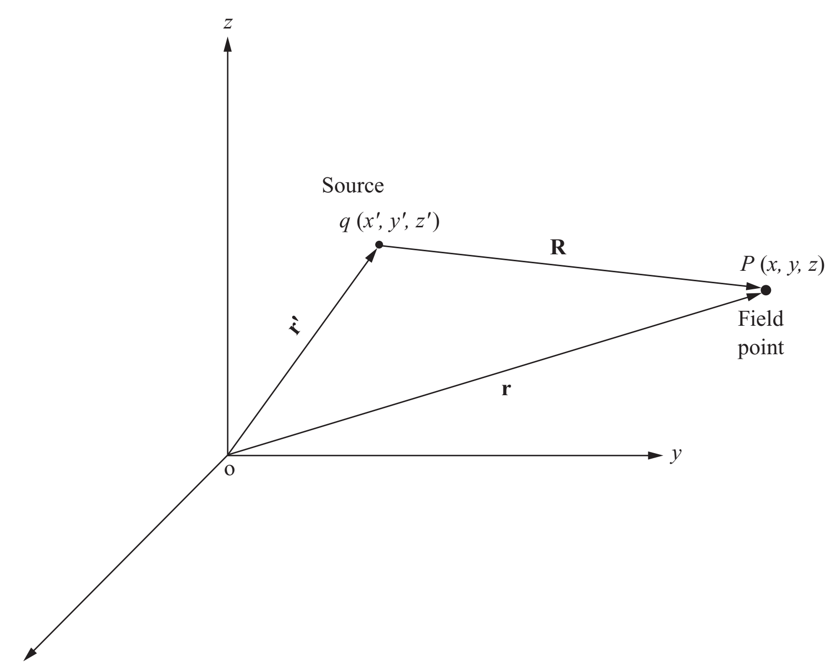

Fig. 1.4 Position vectors of source and field points

volume charge density distribution function. In the limit $\Delta v' \to 0$ we can consider the charge as a point charge and compute the potential at any field point $\mathbf{r}$ due to the charge contained in the volume $\Delta v'$ using the expression given in Eqn (1.39).

$$\Delta V(r,\theta,\phi) = \frac{\rho \Delta v'}{4\pi\epsilon R} \tag{1.40}$$

Let us now consider a time-varying charge $\rho\Delta v'$ with a sinusoidal time variation represented by $e^{j\omega t}$. Heuristically, we can reason out that the effect on the potential due to a change in the charge would travel to the field point with the propagation constant k. Hence for a point charge with an exponential time variation of the form $e^{j\omega t}$, the phase fronts are spherical with the point $\mathbf{r}'$ as the origin. Therefore

$$\Delta V(r,\theta,\phi) = \frac{\rho(x',y',z')\Delta v'}{4\pi\epsilon}\frac{e^{-jkR}}{R} \tag{1.41}$$

The potential at point (r,θ,ϕ) due to a charge distribution $\rho(x',y',z')$ is obtained by integrating Eqn (1.41) over the source distribution

$$V(r,\theta,\phi) = \frac{1}{4\pi\epsilon}\iiint\limits_{V'} \rho(x',y',z')\frac{e^{-jkR}}{R}dv' \tag{1.42}$$

where V' is the volume over which $\rho(x',y',z')$ exists, or the source volume.

The instantaneous value of the scalar potential $\mathcal{V}(r,\theta,\phi,t)$ is obtained by

$$\mathcal{V}(r,\theta,\phi,t) = \mathrm{Re}\Big\{V(r,\theta,\phi)e^{j\omega t}\Big\} = \mathrm{Re}\left\{\frac{1}{4\pi\epsilon}\iiint\limits_{V'} \rho(x',y',z')\frac{e^{-jkR+j\omega t}}{R}dv'\right\} \tag{1.43}$$

Using the relation $v = \omega/k$, this reduces to

$$\mathcal{V}(r,\theta,\phi,t) = \mathrm{Re}\left\{\frac{1}{4\pi\epsilon}\iiint\limits_{V'} \rho(x',y',z')\frac{e^{j\omega(t-\frac{R}{v})}}{R}dv'\right\} \tag{1.44}$$

It is clear from this expression that the potential at time t is due to the charge that existed at an earlier time R/v. Or the effect of any change in the source has travelled with a velocity v to the observation point at a distance R from the source. Therefore, V is also known as the retarded scalar potential.

In Section 1.1.1 it is shown that both, electric scalar potential, V and magnetic vector potential, $\mathbf{A}$, satisfy the wave equation with the source terms being ρ/ϵ and $\mu\mathbf{J}$, respectively. Therefore, a similar heuristic argument can be used to derive the relationship between the current density distribution $\mathbf{J}(x', y', z')$ and the vector potential $\mathbf{A}(r, \theta, \phi)$. Starting from the expression for the magnetic vector potential for a static current density we introduce the delay time $-R/v$ to obtain the retarded vector potential expression for the time-varying current density distribution $\mathbf{J}$. The vector potential at any time t is related to the current density distribution at time $(t - R/v)$. Further, the vector $\mathbf{A}$ has the same direction as the current density $\mathbf{J}$. The relationship between the current density $\mathbf{J}(x', y', z')$ and the vector potential $\mathbf{A}(r, \theta, \phi)$ is given by simply multiplying the static relationship with the e^{-jkR} term. Thus, the retarded vector potential is given by

$$\mathbf{A}(r, \theta, \phi) = \frac{\mu}{4\pi} \iiint\limits_{V'} \mathbf{J}(x', y', z') \frac{e^{-jkR}}{R} dv' \tag{1.45}$$

If the current density is confined to a surface with surface density $\mathbf{J}_s$ (in A/m), the volume integral in the vector potential expression reduces to a surface integral

$$\mathbf{A}(r, \theta, \phi) = \frac{\mu}{4\pi} \iint\limits_{S'} \mathbf{J}_s(x', y', z') \frac{e^{-jkR}}{R} ds' \tag{1.46}$$

For a line current $\mathbf{I}$ (in A), the integral reduces to a line integral

$$\mathbf{A}(r, \theta, \phi) = \frac{\mu}{4\pi} \int\limits_{C'} \mathbf{I}(x', y', z') \frac{e^{-jkR}}{R} dl' \tag{1.47}$$

1.1.3 Solution Procedure

The procedure for computing the fields of an antenna requires us to first determine the current distribution on the antenna structure and then compute the vector potential, $\mathbf{A}$, using Eqn (1.45). In a source-free region, $\mathbf{A}$ is related to the $\mathbf{H}$ field via Eqn (1.13)

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \tag{1.48}$$

and $\mathbf{H}$ is related to the $\mathbf{E}$ field by (Eqn (1.7) with $\mathbf{J} = 0$ in a source-free region)

$$\mathbf{E} = \frac{1}{j\omega\epsilon}\nabla \times \mathbf{H} \tag{1.49}$$

As mentioned in Section 1.1, the computation of the current distribution on the antenna, starting from the excitation, involves solution of an integral equation and is beyond the scope of this book. Here we assume an approximate current distribution on the antenna structure and proceed with the computation of the radiation characteristics of the antenna.

1.2 Hertzian Dipole

A Hertzian dipole is 'an elementary source consisting of a time-harmonic electric current element of a specified direction and infinitesimal length' (*IEEE Trans. Antennas and Propagation* 1983). Although a single current element cannot be supported in free space, because of the linearity of Maxwell's equations, one can always represent any arbitrary current distribution in terms of the current elements of the type that a Hertzian dipole is made of. If the field of a current element is known, the field due to any current distribution can be computed using a superposition integral or summing the contributions due to all the current elements comprising the current distribution. Thus, the Hertzian dipole is the most basic antenna element and the starting point of antenna analysis.

Consider an infinitesimal time-harmonic current element, $\mathbf{I} = \mathbf{a}_z I_0 dl$, kept at the origin with the current flow directed along the z-direction indicated by the unit vector $\mathbf{a}_z$ (Fig. 1.5). I_0 is the current and dl is the elemental length of the current element. Time variation of the type $e^{j\omega t}$ is implied in saying the current element is time-harmonic. Consider the relationship between the current distribution $\mathbf{I}$ and the vector potential $\mathbf{A}$, as shown in Eqn (1.47) and reproduced here for convenience

$$\mathbf{A}(r,\theta,\phi) = \frac{\mu}{4\pi}\int_{C'} \mathbf{I}(x',y',z')\frac{e^{-jkR}}{R}dl' \tag{1.50}$$

Since we have an infinitesimal current element kept at the origin

$$x' = y' = z' = 0 \tag{1.51}$$

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} = \sqrt{x^2+y^2+z^2} = r \tag{1.52}$$

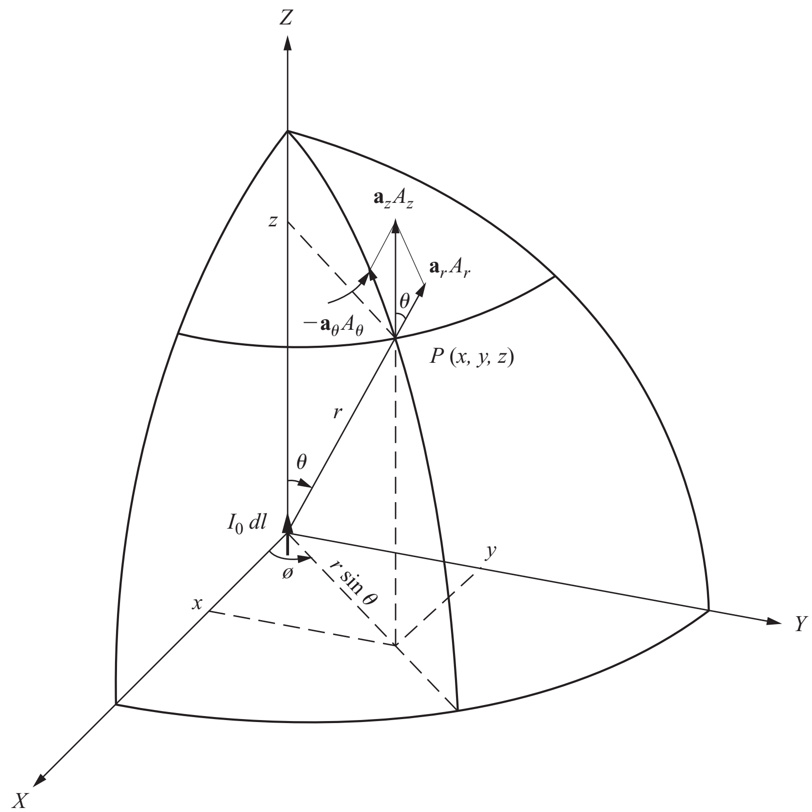

Fig. 1.5 Components of the vector potential on the surface of a sphere of radius r, due to a z-directed current element kept at the origin

Now, the vector potential due to a current element can be written as

$$\mathbf{A}(r, \theta, \phi) = \mathbf{a}_z \frac{\mu}{4\pi} I_0 dl \frac{e^{-jkr}}{r} = \mathbf{a}_z A_z \tag{1.53}$$

Note that the vector potential has the same vector direction as the current element. In this case, the $\mathbf{a}_z$-directed current element produces only the A_z-component of the vector potential.

The **H** and **E** fields of a Hertzian dipole are computed using the relationships given by Eqns (1.48) and (1.49), respectively. The **E** and **H** fields are generally computed in spherical coordinates for the following reasons—(a) the (e^{-jkr}/r) term indicates that the fields consist of outgoing spherical waves which are simple to represent mathematically in spherical coordinates and (b) the spherical coordinate system allows easy visualization of the behaviour of the fields as a function of direction and simplifies the mathematical representation of the radiated fields.

From Fig. 1.5 we can relate the z-component of the vector potential, A_z, to the components in spherical coordinates A_r, A_θ, and A_ϕ as

$$\begin{aligned} A_r &= A_z \cos\theta \\ A_\theta &= -A_z \sin\theta \\ A_\phi &= 0 \end{aligned} \tag{1.54}$$

Taking the curl of $\mathbf{A}$ in spherical coordinates

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta \mathbf{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \tag{1.55}$$

Substituting the components of $\mathbf{A}$ from Eqn (1.54) into Eqn (1.55), we get

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta \mathbf{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_z\cos\theta & -rA_z\sin\theta & 0 \end{vmatrix} \tag{1.56}$$

Since A_z is a function of r alone, its derivatives with respect to θ and ϕ are zero. Hence, the curl equation reduces to

$$\nabla \times \mathbf{A} = \mathbf{a}_\phi \frac{1}{r}\left[\frac{\partial}{\partial r}\left(-rA_z\sin\theta\right) - \frac{\partial}{\partial\theta}(A_z\cos\theta)\right] \tag{1.57}$$

Substituting the expression for A_z and performing the indicated differentiations in Eqn (1.57)

$$\nabla \times \mathbf{A} = \mathbf{a}_\phi \frac{1}{r} A_z \sin\theta\,(jkr+1) \tag{1.58}$$

Substituting the result in Eqn (1.48) and simplifying, we get the expressions for the components of the $\mathbf{H}$ field of a Hertzian dipole in spherical coordinates as

$$H_r = 0 \tag{1.59}$$

$$H_\theta = 0 \tag{1.60}$$

$$H_\phi = jk\frac{I_0 dl \sin\theta}{4\pi}\frac{e^{-jkr}}{r}\left[1 + \frac{1}{jkr}\right] \tag{1.61}$$

The electric field can be obtained from Maxwell's curl equation. Substituting the expression for **H** in Eqn (1.49) we get

$$\mathbf{E} = \frac{1}{j\omega\epsilon}\nabla \times \mathbf{H} = \frac{1}{j\omega\epsilon}\frac{1}{r^2 \sin\theta}\begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta\mathbf{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 0 & 0 & r\sin\theta H_\phi \end{vmatrix} \tag{1.62}$$

Expanding the determinant the equation reduces to

$$\mathbf{E} = \frac{1}{j\omega\epsilon}\frac{1}{r^2 \sin\theta}\left[\mathbf{a}_r \frac{\partial}{\partial\theta}\left(r\sin\theta H_\phi\right) - r\mathbf{a}_\theta \frac{\partial}{\partial r}\left(r\sin\theta H_\phi\right)\right] \tag{1.63}$$

After performing the indicated derivative operations, Eqn (1.63) can be simplified to give the electric field components of a Hertzian dipole in spherical coordinates as

$$E_r = \eta \frac{I_0 dl \cos\theta}{2\pi r}\frac{e^{-jkr}}{r}\left(1 + \frac{1}{jkr}\right) \tag{1.64}$$

$$E_\theta = j\eta \frac{kI_0 dl \sin\theta}{4\pi}\frac{e^{-jkr}}{r}\left(1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right) \tag{1.65}$$

$$E_\phi = 0 \tag{1.66}$$

where $\eta = k/(\omega\epsilon)$ is the intrinsic impedance of the medium.

EXAMPLE 1.5

Show that $\eta = k/(\omega\epsilon)$.

Solution: Substituting $k = 2\pi/\lambda$ and $\omega = 2\pi f$ and simplifying

$$\frac{k}{\omega\epsilon} = \frac{2\pi/\lambda}{(2\pi f)\epsilon} = \frac{1}{(\lambda f)\epsilon}$$

The velocity of the wave, v, is related to the frequency, f, and the wavelength, λ, by $v = f\lambda$, and v is related to the permittivity, ϵ, and permeability, μ, of the medium by $v = 1/\sqrt{\mu\epsilon}$. Substituting these in the above equation and simplifying

$$\frac{k}{\omega\epsilon} = \frac{1}{v\epsilon} = \frac{\sqrt{\mu\epsilon}}{\epsilon} = \sqrt{\frac{\mu}{\epsilon}}$$

The impedance of the medium η is related to ϵ and μ by $\eta = \sqrt{\mu/\epsilon}$ and, therefore

$$\frac{k}{\omega\epsilon} = \eta$$

It is interesting to note that a z-directed current element kept at the origin has only the H_ϕ, E_r, and E_θ components and, further, the fields have components that decay as $1/r$, $1/r^2$, and $1/r^3$, away from the current element. Thus, these expressions form a convenient basis for classifying the fields of any antenna depending on the nature of decay away from the antenna.

To understand the nature of the field behaviour as a function of r, Eqns (1.64) and (1.65) can be re-written as

$$E_r = \eta \frac{k^2 I_0 dl \cos\theta}{2\pi} e^{-jkr} \left(\frac{1}{(kr)^2} + \frac{1}{j(kr)^3} \right) \tag{1.67}$$

$$E_\theta = j\eta \frac{k^2 I_0 dl \sin\theta}{4\pi} e^{-jkr} \left(\frac{1}{kr} + \frac{1}{j(kr)^2} - \frac{1}{(kr)^3} \right) \tag{1.68}$$

A plot of $1/(kr)$, $1/(kr)^2$, and $1/(kr)^3$ as functions of (kr) is shown in Fig. 1.6. For large values of r, i.e., $r \gg \lambda$ or $kr \gg 1$, the terms containing $1/(kr)^2$ and $1/(kr)^3$ decay much faster than $1/(kr)$. Therefore, at large

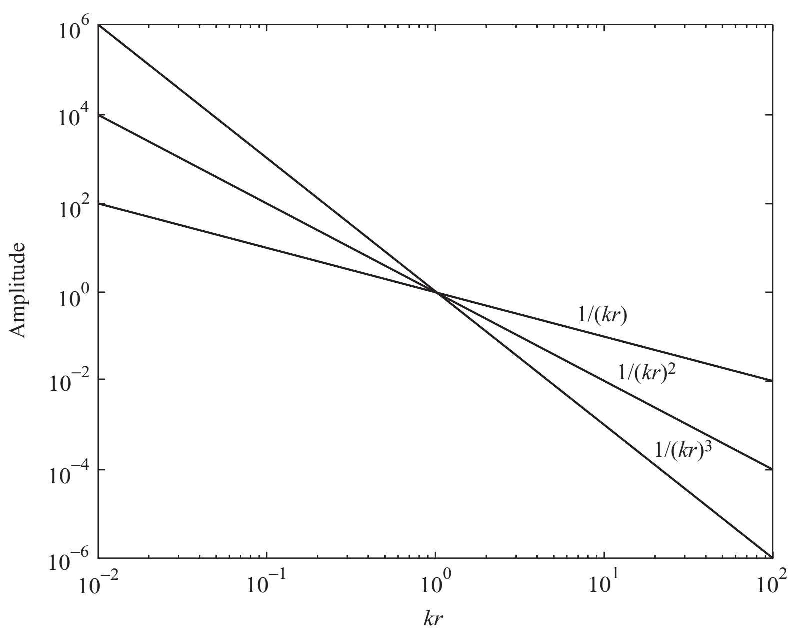

Fig. 1.6 Dependence of $1/(kr)$, $1/(kr)^2$, and $1/(kr)^3$ on (kr)

distances from the Hertzian dipole, only the terms containing $1/r$ are retained in the electric and magnetic field expressions. The electric and the magnetic fields of a z-directed Hertzian dipole for $r \gg \lambda$ are given by

$$E_\theta = j\eta \frac{kI_0 dl \sin\theta}{4\pi} \frac{e^{-jkr}}{r} \tag{1.69}$$

$$H_\phi = j \frac{kI_0 dl \sin\theta}{4\pi} \frac{e^{-jkr}}{r} \tag{1.70}$$

The ratio of E_θ to H_ϕ is equal to the impedance of the medium.

EXAMPLE 1.6

Calculate and compare the r and θ components of the electric field intensities at $x = 100$ m, $y = 100$ m, and $z = 100$ m produced by a Hertzian dipole of length $dl = 1$ m kept at the origin, oriented along the z-axis, excited by a current of $i(t) = 1 \times \cos(10\pi \times 10^6 t)$ A, and radiating into free space.

Solution: The frequency of excitation is $\omega = 10\pi \times 10^6$ rad/s, and therefore $f = 5 \times 10^6$ Hz. The dipole is radiating in free space with parameters $\mu = 4\pi \times 10^{-7}$ H/m and $\epsilon = 8.854 \times 10^{-12}$ F/m. Therefore, the impedance of the medium is

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.854 \times 10^{-12}}} = 376.73\ \Omega$$

and the propagation constant is

$$k = \omega\sqrt{\mu\epsilon} = 10\pi \times 10^6 \sqrt{4\pi \times 10^{-7} \times 8.854 \times 10^{-12}} = 0.1047 \text{ rad/m}$$

The distance r between the field point and the dipole (which is at the origin) is

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{100^2 + 100^2 + 100^2} = 173.2 \text{ m}$$

Using the relation $z = r\cos\theta$, we can compute value of θ as

$$\theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{100}{173.2}\right) = 54.73^\circ$$

Substituting these values in Eqn (1.64)

$$
\begin{aligned}
E_r &= \eta \frac{I_0 dl \cos\theta}{2\pi r} \frac{e^{-jkr}}{r} \left(1 + \frac{1}{jkr}\right) \\
&= 376.73 \frac{1 \times 1 \times \cos 54.73^\circ}{2\pi \times 173.2} \frac{e^{-j0.1047 \times 173.2}}{173.2} \left(1 + \frac{1}{j0.1047 \times 173.2}\right) \\
&= 1.154 \times 10^{-3} (1 - j0.055) e^{-j18.14} \\
&= 1.154 \times 10^{-3} \times 1.002 \angle -3.148^\circ \times 1 \angle 40.65^\circ \\
&= 1.156 \times 10^{-3} \angle 37.51^\circ \text{ V/m}
\end{aligned}
$$

The θ-component of the electric field is evaluated using Eqn (1.65)

$$
\begin{aligned}
E_\theta &= j\eta \frac{kI_0 dl \sin\theta}{4\pi} \frac{e^{-jkr}}{r} \left(1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right) \\
&= j376.73 \frac{0.1047 \times 1 \times 1 \times \sin 54.73^\circ}{4\pi} \frac{e^{-j18.14}}{173.2} \left(1 + \frac{1}{j18.14} - \frac{1}{18.14^2}\right) \\
&= j0.0148 e^{-j18.14} (1 - j0.055 - 3.04 \times 10^{-3}) \\
&= 1 \angle 90^\circ \times 0.0148 \times 1 \angle 40.65^\circ \times 0.9985 \angle -3.158^\circ \\
&= 0.0148 \angle 127.49^\circ \text{ V/m}
\end{aligned}
$$

The wavelength of the EM wave is 60 m and, therefore, at a distance of 2.88λ the θ-component of the electric field is more than 10 times greater than the r-component.

EXAMPLE 1.7

A vector $\mathbf{A}$ can be represented in rectangular coordinate system as $\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$ and in spherical coordinates as $\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi$. Express A_x, A_y, and A_z in terms of A_r, A_θ, and A_ϕ and vice versa.

Solution: The position vector of any point $\mathbf{r}$ in the rectangular coordinate system is given by

$$\mathbf{r} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z$$

From Fig. 1.5

$$
\begin{aligned}
x &= r \sin\theta \cos\phi \\
y &= r \sin\theta \sin\phi \\
z &= r \cos\theta
\end{aligned}
$$

Therefore, the position vector can be written as

$$\mathbf{r} = \mathbf{a}_x r \sin\theta\cos\phi + \mathbf{a}_y r \sin\theta\sin\phi + \mathbf{a}_z r\cos\theta$$

At any point $P(r, \theta, \phi)$, $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ denote the unit vectors along the r, θ, and ϕ directions, respectively. The unit vector along the r-direction is given by

$$\mathbf{a}_r = \frac{\mathbf{r}}{|\mathbf{r}|}$$

where $|\mathbf{r}| = \sqrt{(r\sin\theta\cos\phi)^2 + (r\sin\theta\sin\phi)^2 + (r\cos\theta)^2} = r$ and hence

$$\mathbf{a}_r = \mathbf{a}_x \sin\theta\cos\phi + \mathbf{a}_y \sin\theta\sin\phi + \mathbf{a}_z\cos\theta$$

The unit vector $\mathbf{a}_\theta$ is tangential to the θ-direction. The tangent to the θ-direction is given by $\partial\mathbf{r}/\partial\theta$. Therefore, the unit vector along the θ-direction is given by

$$\mathbf{a}_\theta = \frac{\partial\mathbf{r}/\partial\theta}{|\partial\mathbf{r}/\partial\theta|} = \mathbf{a}_x\cos\theta\cos\phi + \mathbf{a}_y\cos\theta\sin\phi - \mathbf{a}_z\sin\theta$$

Similarly, $\mathbf{a}_\phi$ can be written as

$$\mathbf{a}_\phi = \frac{\partial\mathbf{r}/\partial\phi}{|\partial\mathbf{r}/\partial\phi|} = -\mathbf{a}_x\sin\phi + \mathbf{a}_y\cos\phi$$

This transformation from rectangular to spherical coordinates can be expressed in a matrix form as

$$\begin{pmatrix}\mathbf{a}_r\\ \mathbf{a}_\theta\\ \mathbf{a}_\phi\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{pmatrix}\begin{pmatrix}\mathbf{a}_x\\ \mathbf{a}_y\\ \mathbf{a}_z\end{pmatrix}$$

Equating the rectangular and spherical coordinate representations of $\mathbf{A}$

$$\mathbf{A} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z = \mathbf{a}_r A_r + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi$$

Substituting the expressions for $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ in terms of $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$

$$\begin{aligned}\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z = {} & (\mathbf{a}_x\sin\theta\cos\phi + \mathbf{a}_y\sin\theta\sin\phi + \mathbf{a}_z\cos\theta)A_r\\ & + (\mathbf{a}_x\cos\theta\cos\phi + \mathbf{a}_y\cos\theta\sin\phi - \mathbf{a}_z\sin\theta)A_\theta\\ & + (-\mathbf{a}_x\sin\phi + \mathbf{a}_y\cos\phi)A_\phi\end{aligned}$$

This can be rearranged as

$$\begin{aligned}\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z = {} & \mathbf{a}_x(\sin\theta\cos\phi A_r + \cos\theta\cos\phi A_\theta - \sin\phi A_\phi) \\ & + \mathbf{a}_y(\sin\theta\sin\phi A_r + \cos\theta\sin\phi A_\theta + \cos\phi A_\phi) \\ & + \mathbf{a}_z(\cos\theta A_r - \sin\theta A_\theta)\end{aligned}$$

Equating the coefficients of $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ on both sides, we get a relationship between (A_r, A_θ, A_ϕ) and (A_x, A_y, A_z). This can be represented in matrix form as

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix}$$

The 3×3 matrix is known as the transformation matrix. Let us use the symbol $\mathbf{X}$ to represent the transformation matrix.

The components of $\mathbf{A}$ in spherical coordinates can be written in terms of its components in rectangular coordinates by pre-multiplying both the sides of the above equation by the inverse of the transformation matrix.

$$\begin{aligned}\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} &= \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \\ &= \mathbf{X}^{-1} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}\end{aligned} \tag{1.7.1}$$

The inverse of the transformation matrix is given by

$$\mathbf{X}^{-1} = \frac{1}{\Delta} \begin{pmatrix} \begin{vmatrix} \cos\theta\sin\phi & \cos\phi \\ -\sin\theta & 0 \end{vmatrix} & -\begin{vmatrix} \sin\theta\sin\phi & \cos\phi \\ \cos\theta & 0 \end{vmatrix} & \begin{vmatrix} \sin\theta\sin\phi & \cos\theta\sin\phi \\ \cos\theta & -\sin\theta \end{vmatrix} \\[2ex] -\begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ -\sin\theta & 0 \end{vmatrix} & \begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \cos\theta & 0 \end{vmatrix} & -\begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi \\ \cos\theta & -\sin\theta \end{vmatrix} \\[2ex] \begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi & \cos\phi \end{vmatrix} & -\begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{vmatrix} & \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi \end{vmatrix} \end{pmatrix}^{\mathrm{T}}$$

where Δ is the determinant of the transformation matrix and is equal to unity. On simplifying

$$\mathbf{X}^{-1} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}^{\mathrm{T}}$$

On taking the transpose of the matrix and substituting in Eqn (1.7.1)

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

The transformation matrix has the unitary property, i.e., $\mathbf{X}^{-1} = \mathbf{X}^{\mathrm{T}}$. Using this property we can transform the unit vectors in spherical coordinates into unit vectors in rectangular coodrinates as

$$\begin{pmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a}_r \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{pmatrix}$$

EXAMPLE 1.8

Derive the expressions for the fields of a current element $I_0 dl$ kept at the origin, oriented along the x-axis, and radiating into free space.

Solution: Since the current element is oriented along the x-direction, the magnetic vector potential has only the x-component. Following the procedure given in Section 1.2, we can write the magnetic vector potential as

$$\mathbf{A} = \mathbf{a}_x \frac{\mu_0}{4\pi} I_0 dl \frac{e^{-jkr}}{r}$$

The unit vector $\mathbf{a}_x$ can be written in terms of the unit vectors of the spherical coordinates (see Example 1.7)

$$\mathbf{A} = \frac{\mu_0}{4\pi} I_0 dl \frac{e^{-jkr}}{r} (\mathbf{a}_r \sin\theta\cos\phi + \mathbf{a}_\theta \cos\theta\cos\phi - \mathbf{a}_\phi \sin\phi)$$

The magnetic field is given by

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} = \frac{1}{\mu r^2 \sin\theta} \begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta\mathbf{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

Expanding the determinant

$$\mathbf{H} = \frac{1}{r^2 \sin\theta} \frac{I_0 dl}{4\pi} \left\{ \mathbf{a}_r \left[\frac{\partial}{\partial\theta}\left(-r\sin\theta\sin\phi \frac{e^{-jkr}}{r}\right) - \frac{\partial}{\partial\phi}\left(r\cos\theta\cos\phi\frac{e^{-jkr}}{r}\right)\right] \right.$$
$$- \mathbf{a}_\theta r \left[\frac{\partial}{\partial r}\left(-r\sin\theta\sin\phi \frac{e^{-jkr}}{r}\right) - \frac{\partial}{\partial\phi}\left(\sin\theta\cos\phi\frac{e^{-jkr}}{r}\right)\right]$$
$$\left. + \mathbf{a}_\phi r\sin\theta \left[\frac{\partial}{\partial r}\left(r\cos\theta\cos\phi \frac{e^{-jkr}}{r}\right) - \frac{\partial}{\partial\theta}\left(\sin\theta\cos\phi\frac{e^{-jkr}}{r}\right)\right]\right\}$$

Performing the indicated differentiations and simplifying

$$H_\theta = -j\frac{kI_0 dl}{4\pi}\sin\phi\frac{e^{-jkr}}{r}\left(1+\frac{1}{jkr}\right)$$
$$H_\phi = -j\frac{kI_0 dl}{4\pi}\cos\theta\cos\phi\frac{e^{-jkr}}{r}\left(1+\frac{1}{jkr}\right)$$

The electric field can be calculated from Maxwell's equation $j\omega\epsilon\mathbf{E} = \nabla\times\mathbf{H}$

$$\mathbf{E} = \frac{1}{j\omega\epsilon}\frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta\mathbf{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 0 & rH_\theta & r\sin\theta H_\phi \end{vmatrix}$$

On expanding the determinant, performing the indicated differentiations, and simplifying

$$E_r = \eta\frac{I_0 dl}{2\pi r}\sin\theta\cos\phi\frac{e^{-jkr}}{r}\left(1+\frac{1}{jkr}\right)$$
$$E_\theta = -j\eta\frac{kI_0 dl}{4\pi}\cos\theta\cos\phi\frac{e^{-jkr}}{r}\left(1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right)$$
$$E_\phi = j\eta\frac{kI_0 dl}{4\pi}\sin\phi\frac{e^{-jkr}}{r}\left(1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right)$$

So far we have learnt how to compute the fields due to a current distribution using the vector potential approach. Every antenna can be looked at as a current distribution producing electric and magnetic fields in the surrounding space and, therefore, we have learnt the basics of computing the fields of an antenna. In the following chapter we will learn about properties of antennas and introduce the various terms associated with antennas.

Exercises

1.1 Prove that the spherical coordinate system is orthogonal.

1.2 Show that for any twice differentiable scalar function, ϕ, $\nabla \times \nabla\phi = 0$.

1.3 Show that for any twice differentiable vector function $\mathbf{A}$, $\nabla \cdot \nabla \times \mathbf{A} = 0$.

1.4 Prove the vector identity $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$.

1.5 In a source-free region show that the $\mathbf{E}$ and $\mathbf{H}$ fields satisfy $\nabla^2\mathbf{E} + k^2\mathbf{E} = 0$ and $\nabla^2\mathbf{H} + k^2\mathbf{H} = 0$, respectively.

1.6 Show that $V(r) = V_0 e^{-jkr}/r$, where V_0 is a complex constant and k is a real number, represents a wave travelling in the positive r-direction.

1.7 Plot the equiphase surfaces of the electric field of an EM wave given by (a) $\mathbf{E} = \mathbf{a}_\phi E_0 e^{-jkr}/r$ and (b) $\mathbf{E} = \mathbf{a}_y E_0 e^{-jkx}$, where E_0 is a complex constant.

1.8 Derive Eqns (1.59)–(1.61) from Eqn (1.57).

1.9 Derive Eqns (1.64)–(1.66) from Eqn (1.63).

1.10 For the Hertzian dipole considered in Section 1.2, compute the electric and magnetic fields in the rectangular coordinate system directly by taking the curl of $\mathbf{a}_z A_z$. Now convert the fields into spherical coordinates and compare them with the results given in Eqns (1.59)–(1.61) and Eqns (1.64)–(1.66).

1.11 Show that $\omega\mu = k\eta$, where the symbols have their usual meaning.

1.12 Show that at large distances from a radiating Hertzian dipole ($r \gg \lambda$), the electric and magnetic fields satisfy Maxwell's equations.

1.13 A z-directed Hertzian dipole placed at the origin has length $dl = 1$ m and is excited by a sinusoidal current of amplitude $I_0 = 10$ A and frequency 1 MHz. If the dipole is radiating into free space, calculate the distance in the x-y plane from the antenna beyond which the magnitude of the electric field strength is less than 1×10^{-3} V/m.

Answer: 6279 m

1.14 Derive an expression for the fields of a Hertzian dipole of length dl carrying a current of I_0 which is located at the origin of the coordinate system and oriented along the y-axis.

1.15 Find the strength of the z-component of the electric field at $(0, 100\text{ m}, 0)$ produced by a z-directed Hertzian dipole of length $dl = 0.5$ m, placed at the origin, carrying a current of $i(t) = 2\cos(6\pi \times 10^6 t)$ A, and radiating into free space. If the dipole is oriented along the x-axis, what will be strength of the x-component of the electric field at the same point?

Answer: $E_z = 0.01856\angle{-99.3^\circ}$ V/m; $E_x = 0.01856\angle{-99.3^\circ}$ V/m