Engineering Physics

As per Anna University R17 syllabus

D.K. Bhattacharya
Associate Director
Solid State Physics Laboratory Delhi, DRDO

Poonam Tandon
Associate Professor
Maharaja Agrasen Institute of Technology, New Delhi

Contributions from:
T.K. Subramaniam
Professor of Physics
Sri Sairam Engineering College,Chennai
(Affiliated to Anna University, Chennai)

OXFORD
UNIVERSITY PRESS

OXFORD
UNIVERSITY PRESS

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide. Oxford is a registered trademark of
Oxford University Press in the UK and in certain other countries.

Published in India by
Oxford University Press
Ground Floor, 2/11, Ansari Road, Daryaganj, New Delhi 110002, India

First published in 2017

ISBN-13: 978-0-19-948463-8
ISBN-10: 0-19-948463-5

Typeset in Times New Roman MT Std
by Archetype, New Delhi 110063
Printed in India by Magic International (P) Ltd, Greater Noida

Cover image: valdis torms / Shutterstock

Preface

Science and engineering form the backbone of any technological innovation. Engineering focuses on the conversion of scientific ideas into viable products and technologies. Physics is a fundamental aspect of science. Therefore, knowledge of physics relevant to engineering is critical for converting ideas to products. An understanding of physics also helps engineers comprehend the working and limitations of existing devices and techniques, which eventually leads to new innovations and improvements.

It is interesting to note that in spite of the complexities of modern technology, the underlying principle behind these still remain simple. In fact, it would not be wrong to say that unless the basic physics behind a technology is fully understood, it would be impossible to implement the full potential of the technology.

The fundamental concepts of physics have laid the foundation for advances in engineering technology.

ABOUT THE BOOK

Engineering Physics is primarily designed to serve as a textbook to cater to the requirements of the latest first year engineering physics syllabus of Anna University.

The book thoroughly explains all relevant and important topics in a student-friendly manner. The language and approach towards understanding the fundamental topics of physics is clear. The mathematics has been kept simple and understandable, enabling readers to easily understand the principle and idea behind a concept. The book lays emphasis on explaining the principles using numerous solved examples and well-labelled figures and diagrams. The text is supplemented with plenty of chapter-end practice questions, such as multiple-choice questions, review questions, and numerical problems.

Key Features of the Book and Their Benefits

Features	Benefits
Topical coverage: Topics are arranged as per the latest **R17** syllabus of Anna University.	Completely fulfils the syllabus requirements.

Features	Benefits
Figures and tables: More than **170** well-labelled figures and tables are given.	This will help readers visualize the concepts and principles of physics.
Solved examples: Around **120** solved examples are provided.	This will help readers learn how to apply concepts in a given problem.
List of symbols: A list of symbols is given at the beginning of each chapter.	This facilitates easy referencing of the symbols used in equations and figures across the text.
Summary of concepts, applications, and key formulae: These are given at the end of each chapter.	This helps in quick revision of the important formulae, concepts, and their applications.
Chapter-end self-assessment section: Contains **165** multiple-choice questions, **215** review exercises, and **100+** numerical problems. *Answers to MCQs and numerical problems are given at the end of the book.*	This will help students practice and apply the concepts learnt and also self-check their understanding while preparing for examinations.
Interactive animations: Links for interactive animations, provided as online resources, are indicated by a 'mouse icon' within the text.	These animations will help readers understand the practical implementation of a concept or the occurrence of a phenomenon.

CONTENTS AND COVERAGE

The book has 7 chapters and 3 appendices. The following is a short description of each chapter.

Chapter 1 discusses the different properties of matter, elasticity moduli, and the effect of stress and strain, torsion, twisting couple, and bending moment on bodies. The basic principles and applications of properties of matter are illustrated through torsion pendulum, cantilevers, and I-shaped girders.

Chapter 2 on waves and oscillations elucidates the concept of potential energy, the linear restoring force resulting in linear harmonic oscillations, damped harmonic oscillations, quality factor, and forced vibrations and its phase characteristics. The conventional definition of waves, wave equation, plane progressive waves, characteristics of sound waves, and Doppler effect are discussed.

Chapter 3 discusses the ordered excited state—lasers. It covers the various properties, components, and applications of lasers in detail. The Einstein's transition probabilities have been mathematically derived giving the difference between the three different phenomenon of spontaneous, stimulated emission, and absorption. The chapter also covers various types of lasers, namely, Ruby laser, He-Ne laser, Nd:YAG laser, and semiconductor lasers.

Chapter 4 explains the propagation of light waves in an optical fibre system. It discusses the various types of optical fibres, their classifications, applications, and the losses associated with them. The chapter covers numerical aperture of optical fibre systems, fibre drawings, splicing, LEDs, detectors, fibre optic sensors, and endoscopes.

Chapter 5 introduces the concept of thermal physics and explains the various modes of heat transfer. Concepts of thermal expansion, thermal conductivity, and thermal are elucidated through various applications.

Chapter 6 lays emphasis on quantum physics. The chapter begins with the discussion on black body radiation and Planck's law of blackbody radiation to strengthen the basis of quantum mechanics. It then deals with the Compton effect, concept of matter waves, De Broglie's hypothesis, wave-particle duality, phase and group velocity, Heisenberg's uncertainty principle, and wave function. The chapter presents Schrodinger's time-independent and dependent wave equations to study the quantum or discrete behaviour of particles in a box. Finally, the concept of quantum tunneling and its use in scanning tunneling microscope has been discussed.

Chapter 7 on crystal physics introduces lattices, miller indices, atomic radius, coordination number, and packing factor. Polymorphism, and allotropy are also explained. The different types of crystal structures, crystal imperfections and techniques of growth of single crystals are also covered in detail.

Appendices A, **B**, and **C** covers SI units, important physical constants, and lattice constants respectively.

ONLINE RESOURCES

For the benefit of faculty and students reading this book, additional resources are available online at india.oup.com/orcs/ 9780199484638.

For Faculty

- Solutions manual
- Chapter-wise PPTs

For Students

- MCQs test generator
- Model question papers
- Links to interactive animations (indicated with in the text)

ACKNOWLEDGEMENTS

Dr Bhattacharya is grateful to his family for being extremely cooperative and tolerant during the entire period of developing this book.

Dr Poonam is grateful to Dr Nand Kishore Garg, Chairman, Maharaja Agrasen Society of Education, for his ongoing support and encouragement for creating an academic environment in the pursuit of higher education. She would like to express her gratitude to Prof. B.N. Mishra (Director Emeritus), Sh. Gyanendra Srivastava (CEO), Prof. M.L. Goyal (Director), and Prof. Ram Kishore (HOD, Applied Science) for always motivating and guiding her during the development of this book. She is also thankful to her colleagues and students for their healthy interaction and critical suggestions. Dr Poonam is indebted to her family, without whom this endeavour would not have been possible.

Dr Bhattacharya and Dr Poonam are thankful and indebted to Dr T. K. Subramaniam, Professor of Physics, Department of Humanities and Sciences, Sri Sai Ram Engineering College (affiliated to Anna University), Chennai, for his significant and valuable contributions of content in chapters on Properties of Matter and Thermal Physics.

Last but not the least, Dr Bhattacharya and Dr Poonam would also like to acknowledge with gratitude the support and guidance provided by the editorial team of Oxford University Press, India, without whom the book would not have been a reality. The authors would be grateful to receive suggestions and feedback for the book at dk_bhattacharya@yahoo.co.uk and spuat@yahoo.com.

D.K. Bhattacharya
Poonam Tandon

Detailed Contents

Unit III: Thermal Physics

Road Map to Syllabus

Engineering Physics

(As per Anna University R17 syllabus)

Syllabus	Chapter no.
UNIT I: PROPERTIES OF MATTER	
Elasticity – Stress-strain diagram and its uses - factors affecting elastic modulus and tensile strength – torsional stress and deformations – twisting couple - torsion pendulum: theory and experiment - bending of beams - bending moment –cantilever: theory and experiment – uniform and non-uniform bending: theory and experiment - I-shaped girders - stress due to bending in beams.	**1.** Properties of Matter
UNIT II: WAVES AND FIBRE OPTICS	
Oscillatory motion – forced and damped oscillations: differential equation and its solution – plane progressive waves – wave equation. Lasers: population of energy levels, Einstein's A and B coefficients derivation – resonant cavity, optical amplification (qualitative) – Semiconductor lasers: homojunction and heterojunction. Fibre optics: principle, numerical aperture and acceptance angle - types of optical fibres (material, refractive index, mode) – losses associated with optical fibres - fibre optic sensors: pressure and displacement.	**2.** Waves and Oscillations **3.** Lasers **4.** Fibre Optics
UNIT III: THERMAL PHYSICS	
Transfer of heat energy – thermal expansion of solids and liquids – expansion joints - bimetallic strips - thermal conduction, convection and radiation – heat conductions in solids – thermal conductivity – Forbe's and Lee's disc method: theory and experiment - conduction through compound media (series and parallel) – thermal insulation – applications: heat exchangers, refrigerators, ovens and solar water heaters.	**5.** Thermal Physics
UNIT IV: QUANTUM PHYSICS	
Black body radiation – Planck's theory (derivation) – Compton effect: theory and experimental verification – wave particle duality – electron diffraction – concept of wave function and its physical significance – Schrödinger's wave equation – time independent and time dependent equations – particle in a one-dimensional rigid box – tunnelling (qualitative) – scanning tunnelling microscope.	**6.** Quantum Physics
UNIT V: CRYSTAL PHYSICS	
Single crystalline, polycrystalline and amorphous materials – single crystals: unit cell, crystal systems, Bravais lattices, directions and planes in a crystal, Miller indices – inter-planar distances - co-ordination number and packing factor for SC, BCC, FCC, HCP and diamond structures - crystal imperfections: point defects, line defects – Burger vectors, stacking faults – role of imperfections in plastic deformation - growth of single crystals: solution and melt growth techniques.	**7.** Crystal Physics

CHAPTER 1

Properties of Matter

Learning Objectives

After studying this chapter, students will be able to

- understand the concept of elasticity, stress, and strain
- realize Hook's law and relationship between the different elastic constants
- comprehend the important elastic properties of materials
- learn about tensile strength, torsional stress, deformations, twisting couple, and torsion pendulum
- understand bending moments as well as uniform and non-uniform bending in beams
- describe I-shaped girders and stress due to bending in beams

List of Symbols

a = Distance between the one end of a beam and the weight hanger in metres.
A = Area of the beam $(\mathrm{m})^2$
C = Integration constant
d = Thickness of the beam(scale)
f = Force experienced by a shearing stress
F = Force applied in order to shear a material
g = Acceleration due to gravity
I = Geometric moment of inertia
K = Bulk modulus of elasticity
l = Length of the beam between the knife edges
L = Total length of the beam
m = Mass added to the beam
M = Moment of a force
R = Radius of curvature of the neutral axis
T = Torque of a beam
y = Depression of a beam loaded at the centre or loaded at its free end
α = Longitudinal strain per unit stress
β = Lateral strain per unit stress
σ = Poisson's ratio
σ_s = Bending stress on a material
η = Rigidity modulus of elasticity
ϑ = Frequency
λ = Wavelength

θ = Angular displacement

°C = Degrees Celsius

Δ = Incremental change (in length)

κ = Torsional constant

1.1 INTRODUCTION

All bodies can be "deformed" by suitable applied forces. When the deforming forces are removed, the body is either able to recover its original condition or not able to retain its original condition. As a result, there is a stress in the body. Therefore, the substances or bodies or materials are either elastic or plastic. The elastic modulus determine the extent of change in the condition of a body. Different types of stress, strain and their relationship using the stress-strain diagram. Factors affecting elasticity and tensile strength, and concepts of deformations, twisting couple, and torsion pendulum will be discussed in the chapter.

In this chapter, we will discuss the theory and experiments for bending moment, stress, and uniform and non-uniform bending in beams. Thus, this chapter aims at providing a firm foundation of the basic principles involved in dealing with materials, their properties and applications in engineering.

1.2 ELASTICITY, STRESS, AND STRAIN

Every object tries to oppose any effort/force trying to change its shape and size. The extent of this opposition depends upon its elastic properties. A change in size or shape of a solid body requires the application of an external force. *Any force resulting in a change in shape or size of a body is called a deforming force*. This force can thus change the length, volume, or merely the shape of a body. It, thus, tries to produce a change in the normal equilibrium position of the atoms or molecules constituting the body. The body responds by generating an internal restoring force that resists any change in its shape or size. *The internal restoring force per unit area of a deformed body is called the stress in the body*. Thus,

$$\text{Stress} = \frac{\text{Internal restoring force}}{\text{Area}} \tag{1.1}$$

Two possibilities exist at this stage.

- First, on removal of the deforming force, the restoring force brings the body back to its original shape or size. Such type of bodies are called *elastic bodies.* Thus, this property of materials by virtue of which they tend to regain their original shape and size upon removal of the external or deforming force is called ***elasticity***. For example: a rubber band.
- Second, the body does not regain its original shape or size on removal of the deforming force (or external force). Such type of bodies are called *plastic bodies.* For example, a plastic scale when elongated by applying

a force F, will not regain its original size and shape on removal of the force F.

Hence, two types of bodies or materials exist in nature, namely, elastic bodies and plastic bodies. Bodies that can recover completely their original condition are said to be *perfectly elastic*. For perfectly elastic bodies, the restoring force has got to be equal to the deforming force and expression (1.1) reduces to the following form:

$$\text{Stress} = \frac{\text{Deforming force}}{\text{Area}} \tag{1.2}$$

The SI unit of stress is N/m^2.

On the other hand, bodies that cannot recover to their original shape or size are said to be *perfectly plastic*. Generally, most of the materials which exist in nature are not completely *perfectly elastic* or *perfectly plastic*.

1.2.1 Types of Stress

A deforming force acting normal to the area of a body generates *normal stress* and that acting tangential to the area generates *tangential stress*.

Normal stress can be of two types, namely, *tensile stress* and *compressive stress*.

Tensile stress The restoring force per unit area of a body whose length has been increased in the direction of the deforming force is called tensile stress. This type of stress results from extension produced in a body. A spring gets extended when a mass is at one of its ends. A restoring force generates tensile stress.

Compressive stress The restoring force per unit area of a body whose length has decreased under the application of a deforming force is called *compressive stress.* Springs in the shock absorbers of vehicles experience compressive stress.

Tangential stress (or Shearing stress) The restoring force per unit area of a body whose shape changes due to the application of a tangential force i.e., the force acts along the surface of the body is called *tangential or shearing stress*.

1.2.2 Strain

Let us now turn our attention to the consequences of stresses acting on a body. The deforming force acting on a body leads to a change in shape or size of the body or both. The change produced in the dimensions of a body is reflected in the strain. The fractional deformation that the body undergoes is called *strain*. Thus, *strain is the ratio of change in dimension to the original dimension*. Since it is a ratio it is dimensionless. Thus,

$$\text{Strain} = \frac{\text{Change in dimension}}{\text{Original dimension}} \tag{1.3}$$

There are three types of strain:

1. Linear strain $= \dfrac{\text{Change in length}}{\text{Original length}} = \dfrac{\Delta l}{l}$ (1.4)

2. Volume strain $= \dfrac{\text{Change in volume}}{\text{Original volume}} = \dfrac{\Delta v}{V}$ (1.5)

3. Shape strain or shear = Angular deformation in radians (1.6)

Linear strain and volume strain are easy to visualize. The extension produced in a stretched rubber band is an example of linear strain. The increasing radius of a balloon as it is inflated is an example of volume strain. To visualize shear strain, let us consider a cubical body of side *l* as shown in Fig. 1.1. A tangential force ***F*** is shown to be applied to the face PQRS. An angular deformation *q* is produced in the process, as shown in the figure.

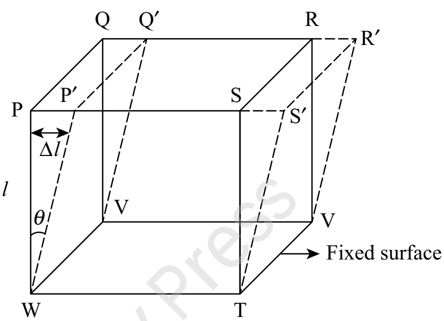


Fig. 1.1 Angular deformation produced by tangential force

The surface PQRS gets shifted to P′Q′R′S′ under the influence of this force. Then,

$$\text{Shearing strain} = \theta = \frac{\Delta l}{l} \tag{1.7}$$

Thus, shearing strain is the angle through which a surface perpendicular to the fixed surface gets angularly displaced.

1.2.3 Hooke's Law

The fundamental law of elasticity was enunciated by Robert Hooke in the year 1679 and it states that *provided the strain is small, the stress is proportional to the strain*. The ratio of stress/strain is a constant (E) and called the *modulus of elasticity* or the *coefficient of elasticity*.

$$\frac{\text{Stress}}{\text{Strain}} = E, \text{ where } E \text{ is an elastic constant} \tag{1.8}$$

Thus, according to Hook's law, stress is directly proportional to the strain produced by it within the elastic limits, i.e., Stress $\propto$ Strain.

Since stress is having the unit of pressure and strain is just a ratio, the units and dimensions of the modulus of elasticity are the same as those of the pressure. The value of E depends on the nature of the material and the conditions it undergoes after it is manufactured.

Note: Elastic limits depicts the maximum stress for which the material is able to recover its original conditions.

1.3 STRESS- STRAIN DIAGRAM AND ITS USES

Suppose we carry out the following experiment. A wire of a uniform cross section is hung vertically from a rigid support. The free end of the wire has a pan on which weights can be kept to subject the wire to different stress levels. Another similar wire is kept close to the first wire. A small weight is attached to the free end of the second wire, and a fixed weight and a Vernier scale is attached to the wire carrying the pan with variable weights. Such a set-up is shown in Fig. 1.2.

Wire A in the figure carries the main scale and wire B the Vernier scale. We now vary the stress by keeping different weights on the pan, and note the corresponding extensions produced using the Vernier and main scale combination. What is the type of curve that we obtain if we plot strain as a function of stress? A typical plot is shown in Fig. 1.3.

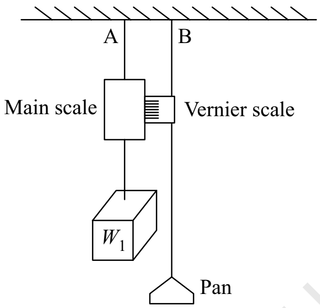


Fig. 1.2 Experimental set-up to study stress–strain relationship

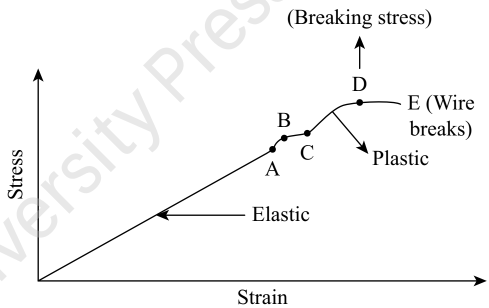


Fig. 1.3 Stress–Strain relationship for a wire

Important points in the curve are clearly identified. Let us now understand the curve in detail. Up to point A, stress and strain are linearly related. In this region, stress is directly proportional to strain. Point A is called the *proportional limit* of the material. After this point, non-linearity sets in for the stress–strain relationship. Up to point B, the wire would return to its original length if the deforming force is removed. B thus represents the *elastic limit*. Beyond point B, the material does not return to its original length on removal of the deforming force. In fact, a residual strain or *permanent strain* remains on removing the force that caused the wire to extend. The material is said to have acquired a *permanent set*. This is the beginning of the plastic region, which continues up to point E. Point C, slightly ahead of point B, is called the yield point of the wire. On increasing the stress further, the strain increases rapidly due to a greatly increased extension of the wire. Point D in the figure represents the maximum of the breaking stress. Here, the material is having the lowest cross-section and is said to be *ductile*. The wire finally breaks or fractures at point E. At this point E, even without the application of any stress, the material simply snips and reaches the lowest part of the graph. Hence, the strain required to completely break the material can be found once we know the details about the breaking point stress.

Materials that elongate considerably and go through a plastic deformation region before breaking are called ductile materials. Wrought iron, lead, and copper are common examples of ductile materials. *Materials that break just beyond the elastic limit are called brittle materials*. High-carbon steel and glass are some examples. There exists another category of material that does not display any region in their stress–strain curve, where stress is directly proportional to strain. These materials, however, show elastic behaviour for a large stress range. Thus, the materials return to its original length on removal of the deforming force; such materials that can be stressed to large strain values are called *elastomers*. We have all handled rubber bands and experienced that they can withstand a large amount of extension. Rubber is a well-known elastomer. The elastic tissue of the large vessel carrying blood from the heart (aorta) is also an elastomer.
Note: Stress and strain are complementary to each other.

1.3.1 Uses of Stress-Strain Curve (or Diagram)

Stress-strain curve is used to measure below properties of materials including metals:

1. Stress-strain curve is used to read the structural load bearing capability or loadability of materials as it is one of the very important properties for materials.
2. It is also useful in finding elastic and plastic deformation limits commonly known as yield point of the material.
3. This curve is used to calculate maximum elongation, maximum tensile strength, reduction in the area of the material, and the fracture or breaking point of the material.

All the aforementioned values are very useful, especially for design engineers, as they enable them to calculate the force, a material can withstand with regard to the cross-section area used without permanent deforming.

1.4 TYPES OF ELASTIC MODULUS (OR ELASTICITY) AND RELATION BETWEEN THEM

Corresponding to the three types of strain learnt in section 1.2.2, we have three types of elasticity or elastic modulus, which are as follows.

1. Linear elasticity called Young's modulus corresponding to linear or tensile strain,
2. Elasticity of volume, or Bulk modulus corresponding to the volume strain, and

3. Elasticity of shape, shear modulus or modulus of rigidity, corresponding to shear strain.

We will be discussing these three types of elastic modulus in this section.

1.4.1 Young's Modulus (Y)

When a deforming force is applied on a body only along a particular direction, the change for unit length in that direction is called *longitudinal or linear or elongation strain*. The force applied per unit area of cross-section is called *longitudinal or linear stress*.

Suppose a force F is applied along the length of a wire l in one direction. Let us also assume the wire has an area of cross-section 'A'. Suppose further that due to the applied stress, the original length 'l' of the wire changed by a magnitude 'Δl'. Then

$$\text{Longitudinal stress} = \frac{F}{A} \tag{1.9}$$

and also,

$$\text{Longitudinal strain} = \frac{\Delta l}{l} \tag{1.10}$$

Within the elastic limit, the ratio of longitudinal stress to the longitudinal strain is called the Young's modulus of elasticity, Y. Therefore,

$$Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}} \tag{1.11}$$

Using Eq. (1.9) and (1.10) in Eq. (1.11) we get

$$Y = \frac{F/A}{\Delta l/l} = \frac{Fl}{(\Delta l)A} \tag{1.12}$$

Eq. (1.12) is the expression for Young's modulus for both compressive and extensive stress. The unit of Young's modulus is N/m^2.

Example 1.1 A load of 3 kg results in an extension of 2 mm in a wire of original length 2 m and diameter 1 m. Calculate the Young's modulus for the material of the wire.

Solution Load, $W = 3 \times 9.8 = 29.4$ N (1.13)

Young's Modulus, Y, is given by,

$$Y = \frac{F}{A} \cdot \frac{L}{\Delta L} \tag{1.14}$$

Putting given values and calculated value of W in Eq. (1.14) we get,

$$y = \frac{29.4 \times 2 \times 4}{(0.002) \times \pi \times (0.001)^2}$$

$$= \frac{29.4\times2\times4\times7}{(0.002)\times22\times(0.001)^2} = 3.74\times10^{10}\ \text{N/m}$$

Example 1.2 A load of 2 kg produces an extension of 4 metres in length and 2 mm of diameter. Calculate the Young's modulus of the wire.

Solution Here, the load applied $W = 2 \times 9.8 = 19.6$ N.

Increase in length = 2 mm = 0.002 m.

Original length = L = 4 m.; radius of the wire = r = 0.001 m.

Using the equation, $\dfrac{F/A}{\Delta l/l}$, we get,

$19.6 \times 4 /3.14 \times (.001)^2 \times .002 = 78.4/0.63 \times 10^{-8} = 1.24 \times 10^{10}$ N/m^2.

Example 1.3 A copper wire of 3 m length and 1mm diameter is subjected to a tension of 5 N. Calculate the elongation produced in the wire, if the Young's modulus of elasticity of copper is 120 GPa.

Solution Young's modulus $Y = \dfrac{F/A}{\Delta l/l}$.

Hence, $\Delta l = F \times l/A$ Y $= 5 \times 3 / [3.14 \times (0.5 \times 10^{-3})^2 \times 120 \times 10^9] = 15.9 \times 10^{-3}$ m.
Therefore, the elongation produced is = 15.9 mm.

Example 1.4 What force is required to stretch a steel wire to double its length when its area of cross-section is 1 cm^2 and Young's modulus of elasticity is $Y = 2 \times 10^{11}$ N/m^2.

Solution The Young's modulus is given by the formula Y = stress / strain = $\dfrac{F.L}{A.l}$,
where, L is original length,
l is increase in length,
A is the area of cross-section, and
F is the force in Newton. Therefore force $F = \dfrac{YAl}{L}$ Newton

Using this formula and substituting the terms, we get, $F = 2 \times 10^7$ Newton

Example 1.5 A wire of length 1 m and radius 0.5 mm elongates by 0.32 mm when stretched by a force of 49 N and twists through 0.4 radian when equal and opposite torques of 3×10^{-3} Nm are applied at its ends. Calculate the elastic constant for iron.

Solution Using the general term, Y = Stress / Strain = $\dfrac{F.L}{A.l}$,

We get, in this problem, $Y = \dfrac{Fl}{\pi r^2 x}$,

Substituting the given values, we get, $Y = 19.5 \times 10^{10}$ N/m^2

1.4.2 Bulk Modulus (K)

The force is applied normally and uniformly to the whole surface of the body, so that, while there is a change of volume, there is no change of shape.

Suppose a body has a volume V and area A. Suppose further that a force F is applied uniformly, normal to the surface area A. If the volume undergoes a change ΔV due to the applied force.

The Bulk modulus, K, is then defined as,

$$K = \frac{F/A}{\Delta V/V} = \frac{FV}{A(\Delta V)} \tag{1.15}$$

The ratio, $\frac{F}{A}$ is equal to the pressure, P. Thus, Eq. (1.15) reduces to,

$$K = \frac{PV}{(\Delta V)} \tag{1.16}$$

Bulk modulus has the unit of N/m^2.

Bulk modulus is sometimes known as "incompressibility" and hence its reciprocal is $1/K$ called compressibility. The unit of compressibility is m^2/N.

1.4.3 Modulus of Rigidity (η)

In this case, while there is a change in the shape of a body, there is no change in its volume. The stress here is a tangential stress. It is clearly equal to the force F divided by the area of the applied force (i.e., F/a).

We will now take the case of a force that does not change the size of a body but only changes the shape of a body. As an example, Fig. 1.4 shows a cube that is deformed in shape due to a tangential force, F.

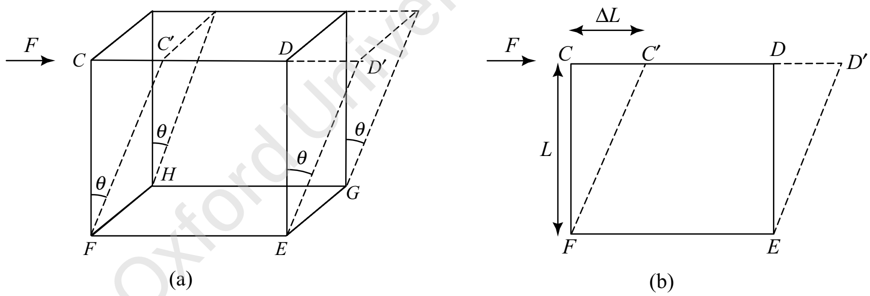


Fig. 1.4 (a) Shearing force on one face of a cube (b) schematic of one face

If 'A' is the area of the force then the tangential stress is given by, F/A. The shearing strain, θ, is given by,

$$\theta \cong \tan\theta \cong \frac{CC'}{CF} \tag{1.17}$$

which can be written as,

$$\theta = \frac{\Delta L}{L} \tag{1.18}$$

where, ΔL is the relative displacement produced by the tangential force and L is the edge length of the cube.

The ratio of tangential stress and the shearing strain is called the modulus of rigidity, η. Using Eq. (1.18) and the expression for tangential stress, we have,

$$\eta = \frac{F/A}{\Delta L/L} = \frac{FL}{A(\Delta L)} \tag{1.19}$$

The unit of modulus of rigidity is N/m^2.

1.4.4 Poisson's Ratio (σ)

When we stretch a string or a wire, it becomes longer but thinner, i.e., the increase in its length is always accompanied by a decrease in its cross-section. In other words a linear or a tangential strain produced in the wire is accompanied by a transverse or a lateral strain of an opposite kind in a direction at right angles to the direction of the applied force.

Hence, within the elastic limit, the lateral strain (β) is proportional to the linear or tangential strain (α) for the material of a given body and the ratio between the two is a constant called "Poisson's ratio" for that material. It is denoted by the letter σ and thus,

$$\text{Poisson's ratio} = \frac{\text{lateral strain}}{\text{linear strain}} = \sigma = \frac{\beta}{\alpha}$$

or $$\sigma = \frac{1}{m} = \frac{\beta}{\alpha}$$

or $$\sigma = \frac{\text{Secondary strain } (\beta)}{\text{Primary strain } (\alpha)}$$

If a force deforms a body in the direction in which it is applied, it also results in deformations in other directions. Let us try to understand using a wire as an example. If the wire is stretched by a force applied along its length, the same force also results in a reduction in its diameter. This is shown schematically in Fig. 1.5.

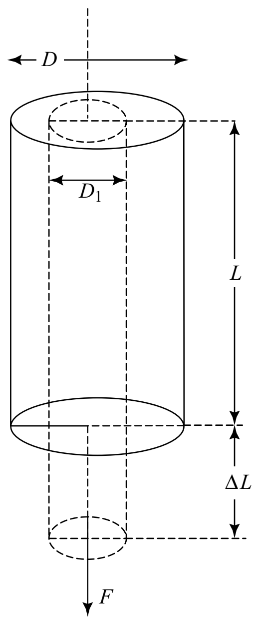


Fig. 1.5 Extension of a wire due to applied force

In this figure a force F extends the length of a wire by a magnitude ΔL and results in a reduction in its diameter from D to D_1. The strain produced along the direction of applied force is called longitudinal strain and that produced in the perpendicular direction is called the lateral strain. The longituding strain per unit stress, α, is given by,

$$\alpha = \Delta L/L \tag{1.20}$$

The lateral strain β is given by,

$$\beta = \frac{(D - D_1)}{D} \tag{1.21}$$

Poisson's ratio, σ, is then defined as,

$$\sigma = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{-(D - D_1)/D}{\Delta L/L}$$

leading to,

$$\sigma = -\frac{(D - D_1)L}{D(\Delta L)} \tag{1.22}$$

Here, the negative sign is used due to the fact that the longitudinal strain and the lateral strain have the opposite sense, i.e., if the length increases then the diameter of the wire decreases and vice versa.

1.4.5 Relation between Elastic Modulus (Qualitative)

We define α as the increase per unit length per unit tension along the direction of a force *F*. (Here, α is linear or tangential strain). We define β as the contraction produced per unit length per unit tension in a direction perpendicular to the force. (Here, β is lateral strain) *K* is the Bulk modulus, *Y* is the Young's modulus, and η is the rigidity modulus of a material.

$$K = \frac{1}{3(\alpha - 2\beta)} \quad \text{(Relation I)}$$

or $$\alpha - 2\beta = 1/3K \tag{1.23}$$

$$\eta = \frac{1}{2(\alpha + \beta)} \quad \text{(Relation II)}$$

or $$\alpha + \beta = \frac{1}{2\eta} \tag{1.24}$$

Subtracting Eq. (1.23) from Eq. (1.24), we get

$$\overset{(-)}{\alpha} \overset{(+1)}{-\,2\beta} = \overset{(-1)}{\frac{1}{3K}}$$

$$\alpha + \beta = \frac{1}{2\eta}$$

$\therefore$ (1.24) – (1.25) gives,

$$3\beta = \frac{1}{2\eta} - \frac{1}{3K} = \frac{3K - 2\eta}{6\eta K}$$

or $$\beta = \frac{3K - 2\eta}{18\,\eta K}$$

Multiplying Eq. (1.24) by 2 and adding it to Eq. (1.23) we get,

$$3\alpha = \frac{1}{\eta} + \frac{1}{3K} \quad \text{or} \quad \alpha = \frac{3K + \eta}{9\eta K}$$

or $$\frac{1}{Y} = \frac{3K + \eta}{9\,\eta K} \quad \text{or} \quad Y = \frac{9\,\eta K}{3K + \eta}$$

i.e., $\frac{9}{Y} = \frac{3K+\eta}{\eta K}$ or $\frac{9}{Y} = \frac{3}{\eta} + \frac{1}{K}$

Relation between the different elastic constants and Poisson's ratio is discussed separately in the following sections.

Relation between *Y* and *β*

Let us consider a cube of unit side length. Let us further assume a unit force acts along one of the sides of the cube. The Young's modulus, Y, is given by

$$Y = \frac{\text{Stress}}{\text{Longitudinal strain}} = \frac{1}{\alpha} \tag{1.25}$$

Relation between *K*, *α*, and *β*

Let us take a cube of unit length with the origin coinciding with the origin 'O'. The three sides of the cube are assumed to be parallel to the three mutually perpendicular axes as shown schematically in Fig. 1.6.

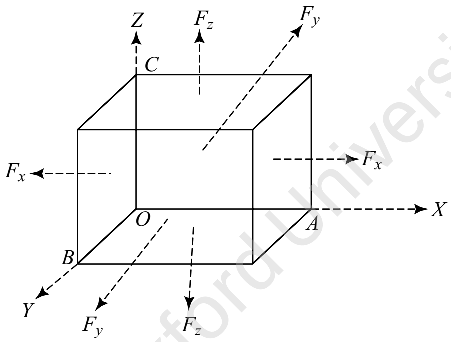


Fig. 1.6 Forces applied on a cubical body

Let α represent the increase in length per unit length per unit force along the direction of the applied force. Also, let β represent the increase in length per unit length per unit force in a direction that is perpendicular to the applied force. Forces F_x, F_x; F_y, F_y and F_z, F_z represent pairs of forces applied along x, y, and z directions respectively. Lengths OA, OB and OC after application of the forces have the expressions,

$$OA = 1 + \alpha F_x - \beta F_y - \beta F_z \tag{1.26}$$

$$OB = 1 + \alpha F_y - \beta F_x - \beta F_z \tag{1.27}$$

and

$$OC = 1 + \alpha F_z - \beta F_x - \beta F_y \tag{1.28}$$

Volume of the cube is,

$$OA \times OB \times OC \tag{1.29}$$

Putting OA, OB, OC from Eqs (1.26), (1.27), (1.28) into Eq. (1.29) yields,

$$\text{Volume} = (1 - \alpha F_x - \beta F_y - \beta F_z)(1 + \alpha F_y - \beta F_x - \beta F_z)$$
$$(1 + \alpha F_z - \beta F_x - \beta F_y) \tag{1.30}$$

Neglecting higher powers of α, β, Eq. (1.30) can be simplified to,

$$\text{Volume} = 1 + (\alpha - 2\beta)(F_x + F_y + F_z) \tag{1.31}$$

If, $F_x = F_y = F_z = F$, Eq. (1.31) reduces to,

$$\text{Volume} = 1 + (\alpha - 2\beta)3F \tag{1.32}$$

Original volume = 1

$$\text{Change in volume} = (\alpha - 2\beta)3F \tag{1.33}$$

Using Eq. (1.33),

$$\text{Volume strain} = \frac{3F(\alpha - 2\beta)}{1} \tag{1.34}$$

$$\text{Bulk modulus, } K = \frac{\text{Stress}}{\text{Volume strain}} \tag{1.35}$$

Using Eq. (1.34) in Eq. (1.35), we get,

$$K = \frac{F}{3F(\alpha - 2\beta)} = \frac{1}{3(\alpha - 2\beta)} \tag{1.36}$$

Eq. (1.36) gives the relationship between K, α and β.

Relation between *Y*, *K* and α

Equation (1.36) can rewritten as,

$$K = \frac{1/\alpha}{3(1 - 2\beta/\alpha)} \tag{1.37}$$

Also, $$\sigma = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{\beta}{\alpha} \tag{1.38}$$

Using Eq. (1.35) and (1.36) in Eq. (1.37), we get,

$$K = \frac{Y}{3(1 - 2\sigma)} \tag{1.39}$$

Equation (1.39) gives a relation between K, Y, and σ.

Relation between η, α, and β

Let us take a cube of side length L. A tangential stress S is applied on the cube as shown in Fig. 1.7 for a face of the cube.

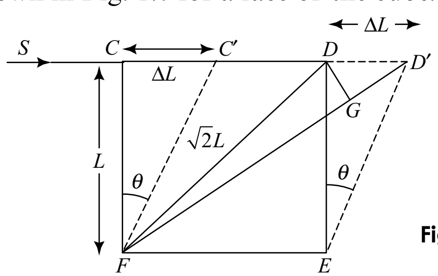


Fig. 1.7 Tangential stress *S* on a face of a cube

As a consequence of the applied stress the cube gets distorted to $C'D'EF$ from the original $CDEF$. The diagonal CE contracts and the diagonal FD expands through the same magnitude. DG is perpendicular on the new diagonal FD'. The tangential stress S is equivalent to a compressive stress S along the diagonal CE and a tensile stress along the diagonal FD. Let α represents the longitudinal strain per unit stress and β represents the lateral strain per unit stress.

From Fig. 1.7 we can see that,

$$\text{Strain} = \frac{GD'}{FD} = S(\alpha + \beta) \tag{1.40}$$

Also,

$$GD' = DD'\cos(DD'G) = DD'\cos GS = \frac{(\Delta L)}{\sqrt{2}} \tag{1.41}$$

and,

$$FD = \sqrt{L^2 + L^2} = \sqrt{2}L \tag{1.42}$$

Using Eqs. (1.41) and (1.42) in Eq. (1.40), we get

$$S(\alpha + \beta) = \frac{(\Delta L)/\sqrt{2}}{\sqrt{2}L} = \frac{(\Delta L)}{2L} \tag{1.43}$$

For small θ, $\frac{(\Delta L)}{L} \approx \theta$, thus Eq. (1.43) reduces to,

$$S(\alpha + \beta) = \frac{\theta}{2}$$

which can be written as,

$$\frac{S}{\theta} = \frac{1}{2(\alpha + \beta)} \tag{1.44}$$

But $\frac{S}{\theta} = \eta =$ modulus of rigidity. Thus Eq. (1.44) can be rewritten as,

$$\eta = \frac{1}{2(\alpha + \beta)} \tag{1.45}$$

leading to,

$$\eta = \frac{1/\alpha}{2(1 + \beta/\alpha)} \tag{1.46}$$

Using Eq. (1.25) and (1.38) in Eq. (1.46), we get

$$\eta = \frac{Y}{2(1 + \sigma)} \tag{1.47}$$

Equation (1.47) gives a relation between Y, η and S.

Relation between *Y*, *K*, η and σ

From Eq. (1.39) we have

$$K = \frac{y}{3(1 - 2\sigma)}$$

which can be rewritten as,

$$\frac{Y}{3K} = 1 - 2\sigma \tag{1.48}$$

Also, from Eq. (1.47) we get,

$$\frac{Y}{\eta} = 2(1 + \sigma) \tag{1.49}$$

Adding Eq. (1.48) and (1.49), we get

$$\frac{Y}{3K} + \frac{Y}{\eta} = 3$$

yielding,

$$\frac{1}{3K} + \frac{1}{\eta} = \frac{3}{Y}$$

giving,

$$Y = \frac{9\eta K}{3K + \eta} \tag{1.50}$$

Eq. (1.50) is a relation between Y, η and K.

Dividing Eq. (1.48) by Eq. (1.49) results in,

$$\frac{\eta}{3K} = \frac{1 - 2\sigma}{2(1 + \sigma)}$$

giving,

$$2\eta\,(1 + \sigma) = 3K(1 - 2\sigma)$$

yielding,

$$2\eta + 2\eta\sigma = 3K - 6k\sigma$$

which can be rewritten as,

$$2\eta\sigma + 6K\sigma = 3K - 2\eta$$

leading to,

$$\sigma = \frac{3K - 2\eta}{3(\eta + 3K)} \tag{1.51}$$

Eq. (1.51) is a relation between σ, K and η.

Example 1.6 A wire of length 1 m and diameter 1 mm is fixed at one end and a couple is applied at the other end so that the wire twists by $\pi/2$ radians. Calculate the moment of the couple required if rigidity modulus of the material is $\eta = 2.8 \times 10^{10}$ N/m^2.

Solution The required formula is, restoring couple is $T = \dfrac{\pi\eta\theta}{2l} r^4$

Substituting the given values, we get, $T = 4.3 \times 10^{-3}$ Nm.

1.4.6 Factors affecting Elastic Modulus (or Elasticity)

The elastic properties of a material are linked up with the fine mass of its structure. Single crystals, when subjected to deformation, show a remarkable increase in their hardness. For example, a single crystal of silver shows a remarkable increase in hardness on being stretched to more than twice its length. Its stiffness increases to as much as ninety-two times its original strength. Thus, the factors affecting elasticity are:

1. Effect of hammering, rolling and annealing;
2. Effect of impurities;
3. Effect of change of temperature

Effect of hammering, rolling, and annealing Operations like hammering and rolling, etc. help break up the crystal grains into smaller unit results in an increase or extension of their elastic properties, whereas operations like annealing (ie, heating and then cooling gradually) tends to produce a uniform pattern of orientation of the constituent crystals, by orienting them all in one particular direction and thus forming larger crystal grains, resulting in a decrease in their elastic properties or an increase in softness or plasticity of the material.

Effect of impurities It is well known that sometimes suitable impurities are deliberately added to metals to help bind their crystal grains better, without affecting their orientation. For example, carbon and potassium are added in minute quantities to molten iron and gold respectively for this purpose. Such impurities naturally affect the elastic properties of the metal to which they are added, enhancing or impairing them. In either case, the elastic properties are considerably strengthened.

Effect of temperature A change in temperature also affects the elastic properties of a material—rise in temperature usually decreases its elasticity and vice-versa. Exceptions are "invar steel" whose elasticity remains practically unaffected by any changes in temperature.

Thus, lead becomes quite elastic and rings like steel when struck by a wooden mallet, and cooled in liquid air.

Carbon filament, which is highly elastic at the ordinary temperature, becomes plastic when heated by the current passed through it, as a result, it can be easily distorted by a magnet brought near it.

1.5 TENSILE STRENGTH

Tensile strength is the maximum tensile force required to pull something like a rope, wire, structural beam, etc. to the point where it breaks. It is measured as the maximum stress that a material can take without getting fractured on when being stretched divided by the original cross-sectional area of the material. Tensile strength specifies the point when a material goes from elastic to plastic deformation.

Tensile strength can classified into three types:

1. Yield strength – It is the stress that a material can withstand without permanent deformation.

2. Ultimate strength – It is the maximum stress a material can withstand.
3. Breaking strength – It is the stress coordinate on the stress-strain curve at the point of fracture (or rapture).

This concept is generally applied in various engineering fields such as material science, mechanical engineering, textile engineering, and structural engineering.

The working stress on a material should be kept below the tensile stress in order to prevent the material to lose its elastic properties. Let us consider a small cube (EFGD) of length *l*, as shown below. The centre of the cube is H and the lower part of the cube EF is clamped. Let a tangential force be applied (F) to the cube along the positive *x*-axis direction. The stress *T* is calculated as force per unit area,

i.e., $$\frac{F}{l^2} = T \tag{1.52}$$

and it acts along the direction DG. The opposite reaction to this force is faced by the bottom surface EF in a direction towards negative *x*-axis .Thus the two faces have equal and opposite forces that constitute a deflecting couple. This couple tends to pull the object in the clockwise direction. The restoring force of equal magnitude, but opposite in direction, tends to act in the directions along FG and DE to balance this deflecting couple along DG and EF. The force that acts along DG can be made to resolve along DH and GH into two equal components *f*.

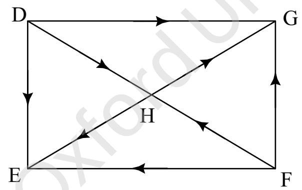


Fig. 1.8 Tangential force F (shear) acting on a surface of a cube EFGD

Similarly, the other three forces can be resolved into smaller components, each with magnitude *f*. So, in effect, we have four forces, each of value f, acting tangentially on the faces of the cube and these forces are resolve into eight forces, each of value *f*, acting along the diagonals DF and GE. The force acting along the diagonal DF tends to compress the square while the force along the diagonal GE tends to extend the square. Hence, an area of the triangular face, DFG, is generated .The area of this triangle is equal to

$$l \times 1\sqrt{2} = l^2\sqrt{2} \tag{1.53}$$

The force acting on either side of the normal to it is 2*f* at the centre H,

therefore, the compressive stress $= \dfrac{\text{force}}{\text{area}} = \dfrac{2f}{l^2\sqrt{2}}$ (1.54)

But, $f = \text{F} \cos 45° = F/\sqrt{2}$ (1.55)

Hence, compressive stress is equal to $\dfrac{F}{l^2} = T$ (1.56)

which is the tangential stress.

Similarly, it can be proved that the tensile stress along GE is also equal to tangential stress *T*. Thus, a shearing stress is equal to a linear tensile stress and to an equal linear compressive stress.

As tensile strength is a limit state of tensile stress, it results in (a) ductile failure i.e., a material, if ductile, that has already begun to flow plastically rapidly forms a constricted region called a neck, where it then breaks and, (b) brittle failure, i.e., the material suddenly breaks in two or more pieces at a low stress state. Tensile strength can be used in terms of either true stress or engineering stress.

1.5.1 Factors affecting Tensile Strength

The following factors affect the tensile strength of a material:

Effect of temperature Tensile strength and temperature are inversely proportional. Decrease in temperature causes an increase in the tensile strength and yield strength of all metals while tensile strength decreases with rise in temperature.

Effect of grain size The metals are composed of crystals or grains. If the grain size of a metal is small, it is called a fine grained metal, on the other hand, when the grain size is comparatively large, then it is called a coarse grained metal. A fine grained metal has a greater tensile and fatigue strength. It can be easily hardened.

Effect of heat treatment Heat treatment is combination of heating and cooling applied to a material. Mechanical properties like tensile strength, toughness and shock resistance can be improved by heat treatment.

Effect of impurities Presence of different impurities in the material affects tensile strength differently. Commonly found impurities are C, Cr, Ni, Mo, V, Nb, Ti, etc.

1.6 TORSIONAL STRESS AND DEFORMATIONS

Torsion is the twisting of an object due to an applied torque. Torque is a rotating force capable of turning a body. A stress is an internal resistance offered by a body per unit area of the cross section. For studying torsional stress, we may simply define it in terms of shearing stress which is produced when we apply the twisting moment to the end of a shaft about its axis. For example, when we turn a screw driver to produce torsion, our hand applies a torque '*T*' to the handle and twists the shank of the screw driver.

1.6.1 Deformations

Deformation means change in the shape or dimensions of a body as a result of stress and strain on a material. Let us understand this by taking an example of a shaft attached to the wall and rotating it, as shown in Fig. 1.9.

From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length η.

$\varphi \propto T$ and

$\varphi \propto L$, where φ is the shearing angle, T is the torque applied and L is the length of the material.

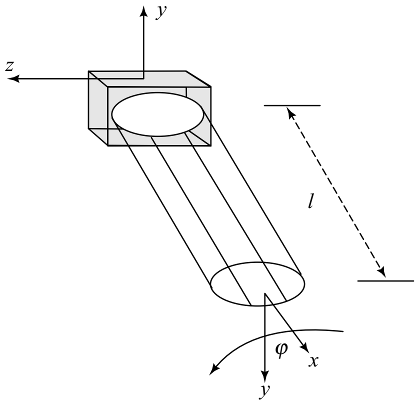


Fig. 1.9 Shaft with a twisted-rotational torque

A circular shaft remains undistorted because its axis is symmetric about the centre. A non-circular shaft, on the other hand, when subjected to torsion, will be distorted, because it is not having an axis that is symmetrical about its centre. For any type of circular shafts, whether it is a solid material or a hollow material, a circular shaft will remain undistorted due to torsion.

1.6.2 Twisting Couple

A pair of forces F, equal in magnitude, but oppositely directed, and displaced by perpendicular distance constitute a *cou*ple. It can also be defined as a system of forces with a resultant moment but without any force acting on it. The resultant moment of a couple is called as *torque*.

Twisting Couple of a Cylindrical Object Let us consider a cylindrical object subjected to torsion. This cylinder is having length l metres and let R be the radius of the cylinder. Since the cylinder is subject to torsion, which is essentially a rotation at the movable end while nothing happens to the fixed end of the cylinder, a twisting couple is accompanied by a restoring couple inside the cylinder. It is required to imagine that this cylinder consists of many coaxial cylinders and one such cylinder is having radius s and thickness ds as shown in Fig. 1.10. Let GH be a parallel line to the central axis EF and now, when the cylinder is twisted, the line GH is twisted through an angle φ, so that the shearing angle is GFC.

From the diagram HC is $= s\theta = l\varphi$ or $\varphi = \dfrac{s\theta}{l}$;

Rigidity modulus $\eta = \dfrac{\text{Shearing stress}}{\text{Shearing strain}}$

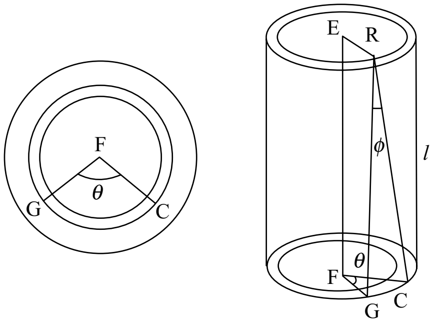


Fig. 1.10 Twisting couple of a cylinder

Hence, $\eta \times$ shearing strain $= \eta\varphi = \dfrac{\eta s\theta}{l}$ (1.57)

But shearing stress $= \dfrac{\text{Shearing force}}{\text{Area over which the force acts}}$

Shearing force = shearing stress × area over which the force acts. But the area over which the force acts = $\pi (s + ds)^2 - \pi s^2$

This area is equal to $\pi x^2 + 2\pi s ds + \pi ds^2 - \pi s^2$ (ds^2 term is neglected since it is very small). Thus, we get,

Area over which force acts = $2\pi s ds$.

The shearing force

$$F = \frac{\eta s\theta}{l} \times 2\pi s ds = \frac{2\pi\eta\theta}{l} s^2 ds \quad (1.58)$$

The moment of the force about the axis EF of the cylinder Force × Perpendicular distance.

$$\frac{2\pi\eta\theta}{l} s^2 ds \times s = \frac{2\pi\eta\theta}{l} s^3 ds \quad (1.59)$$

The moment of this force on the entire cylinder of radius r is obtained by integrating the Eq. (1.59) between the limits $s = 0$ to $s = r$.

Hence, the twisting couple,

$$T = \int_0^r \frac{2\pi\eta\theta}{l} s^3 ds$$

i.e., $$T = 2\pi\eta\theta = \left[\frac{s^4}{4}\right]_0^r$$

Applying the limits, we have,

$$T = \frac{\pi\eta\theta}{2l} r^4 \quad (1.60)$$

In the above expression, if $\theta = 1$ radian, then, we get,

Twisting couple per unit twist $T = \dfrac{\pi\eta}{2l} r^4$

This is the twisting couple required to produce a twist of unit radian in a cylinder is called as *torsional rigidity* for a material of the cylinder.

Special case In the hollow cylinder case, we have, for a cylinder of length l and inner radius r_1 and outer radius r_2, twisting couple of the cylinder

$$T' = \int_{r_1}^{r} \left[\frac{2\pi\eta\theta}{l}\right] s^2 \mathrm{d}s \; = (r_2{}^4 - r_1{}^4)$$

and hence the couple per unit twist of the cylinder = $T' = \dfrac{\pi\eta}{2l}(r_2{}^4 - r_1{}^4)$, assuming $\theta = 1$ radian.

1.7 TORSION PENDULUM: THEORY AND EXPERIMENT

Theory A torsional pendulum consists of a rigid object (like a circular disc) suspended by a wire attached to a rigid support as shown in the Fig. 1.11.

When this rigid object is twisted through an angle θ, the twisted wire exerts a restoring torque on the object which will be proportional to angular displacement θ.

Here, $T = -\kappa\theta$ (1.61)

where κ is called as the torsional constant of the supporting wire. The value of κ can be found by applying a known value of torque T in order to twist the wire through an angle θ. Using Newton's second law of motion for rotational motion, we have,

$$T = -\kappa\theta = Id^2\theta/dt^2 \quad (1.62)$$

Hence, $= -\mathrm{d}^2\theta/\mathrm{d}t^2 = -\kappa/I \cdot \theta$ (1.63)

This can be seen as the same result of the simple harmonic oscillator problem, where we had,

$$\omega = \sqrt{\kappa/I}$$

and the period of the simple pendulum was

$$T = 2\pi\sqrt{I/\kappa} \quad (1.64)$$

where, I = moment of inertia of the suspended body,

κ = couple/unit twist

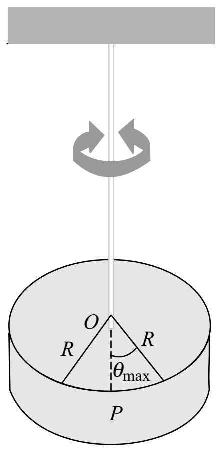


Fig. 1.11 Torsional pendulum

This system is known as the *torsional pendulum*. As long as the elastic limit of the wire is not exceeded, there is no limit to the small-angle restriction.

Since, we have an expression for couple per unit twist as,

$$\kappa = \frac{1}{2}\frac{\pi\eta R^4}{l} \quad (1.65)$$

Where, l = length of the suspension wire; R = radius of the wire; η = rigidity modulus of the suspension wire.

Substituting Eq. (1.65) in Eq. (1.64) and squaring, we get an expression for rigidity modulus for the suspension wire as,

$$\eta = \frac{8\pi I l}{T^2 R^4} \tag{1.66}$$

We can use Eq. (1.66) to calculate the moment of inertia of the disc, i.e.,

$$I = (1/2)\, MR^2$$

Now, let us consider, I_0 be the moment of inertia of the disc alone and I_1 & I_2 be the moment of inertia of the disc with identical masses at distances d_1 & d_2 respectively (refer to Fig. 1.12). If I' is the moment of inertia of each identical mass about the vertical axis passing through its centre of gravity, then

$$I_1 = I_0 + 2I' + 2md_1^2 \tag{1.67}$$

$$I_2 = I_0 + 2I' + 2md_2^2 \tag{1.68}$$

Thus, on subtracting Eq. (1.68) from Eq. (1.67), we get,

$$I_2 - I_1 = 2m(d_2^2 - d_1^2) \tag{1.69}$$

Using Eq. (1.64), we get,

$$T_0^2 = 4\pi^2 \frac{I_0}{\kappa} \tag{1.70}$$

$$T_1^2 = 4\pi^2 \frac{I_1}{\kappa} \tag{1.71}$$

$$T_2^2 = 4\pi^2 \frac{I_2}{\kappa} \tag{1.72}$$

On subtracting Eq. (1.72) from Eq. (1.71), we get,

$$T_2^2 - T_1^2 = 4\pi^2 \frac{1}{\kappa}(I_2 - I_1) \tag{1.73}$$

where, T_0, T_1, T_2 are the periods of torsional oscillation without identical mass, with identical pass at position d_1, d_2 respectively.

On dividing Eq. (1.70) by Eq. (1.73) and using Eq. (1.69),

$$\frac{T_0^2}{\left(T_2^2 - T_1^2\right)} = \frac{I_0}{(I_2 - I_1)} = \frac{I_0}{2m\left(d_2^2 - d_1^2\right)} \tag{1.74}$$

Therefore, the moment of inertia of the disc,

$$I_0 = 2m(d_2^2 - d_1^2)\frac{T_0^2}{\left(T_2^2 - T_1^2\right)} \tag{1.75}$$

Now substituting Eq. (1.75) in Eq. (1.66), we get the expression for rigidity modulus 'η' as,

$$\eta = \frac{16\pi m\left(d_2^2 - d_1^2\right)}{R^4}\frac{l}{(T_2^2 - T_1^2)}(\mathrm{N/m^2}) \tag{1.76}$$

Experiment The aim of the experiment is to find the modulus of rigidity of a given wire. *Rigidity modulus* is defined as the ratio of shearing stress to the shearing strain.

$$\eta = \text{Shearing Stress/ Shearing Strain}$$

$$\eta = \frac{F/A}{\theta} \quad (\text{N/m}^2)$$

Here, F is the stress which is force exerted on the object, A is the area of the object under stress, and θ is the shearing strain (angle). The shearing strain is also calculated by measuring the ratio of the horizontal distance (Δx) that the shearing face moves, to the height of the object (h). Strain is given by $\left(\frac{\Delta x}{h}\right)$.

Figure 1.12 shows the experimental set-up of torsional pendulum.

In the first part of the experiment, we have to find the 'moment of inertia' I_g of the circular solid disc which is suspended about the vertical axis. This is done by the method of torsional oscillations. The torsion pendulum is made to suspend about the vertical axis and its height l metre, is measured from the fixed end (wall) to the point of suspension O. A stop watch is taken and the torsional pendulum is made to oscillate by gently giving a twist to the wire, so that the pendulum oscillates about the mean position P. The time taken for ten or twenty oscillations is observed as T and hence, the time taken for one oscillation is calculated as T_0 (seconds). This is also known as 'time period' of oscillation of the torsional pendulum.

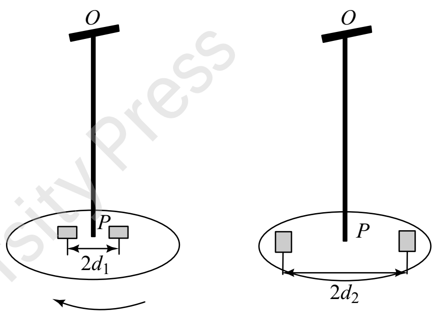


Fig. 1.12 Experimental set-up of torsional pendulum

Table for recording time period for10/20 oscillations

Sl. no.	Length of the torsional pendulum l (in m)	Time taken for 10/20 oscillations (seconds)	Time period			$\frac{T_0^2}{(T_2^2 - T_1^2)}$
			T_0	T_1	T_2	
1	l_1					
2	l_2					
3	l_3					

Two equal masses of either 50 gm each or 100 gm each are taken and these are kept on top of the circular disc. In the first case, the masses are kept very close to the string or wire attached and the distance between them is measured as $2d_1$ cm. It is then divided by two, so that we get the distance from any one of the masses to the wire, when kept at the closest position. The time period of oscillation for this added mass is now calculated, by gently turning the system from its neutral position at P and by creating torsional oscillations. The total time taken for ten or twenty oscillations is observed and hence the time taken for one oscillation, namely T_1 seconds, is noted down. Then, the two masses of 50 gm or 100 gm each are placed at the farthest end of the circular disc, without making the masses falling down, and the distance between them is measured as 2d$_2$ cm. It is then divided by two, so that the distance of any one of the masses from the farthest end to the wire is now known. The system is now made to oscillate about its mean position P, and the total time taken for ten or twenty oscillations, using the stop watch, is observed. Then the 'time period' is calculated to be T_2 seconds. This experiment is repeated for three different lengths, l_1, l_2, and l_3 lengths of the torsional pendulum. Knowing the masses added, on any one side, in kilograms, we can calculate the moment of inertia of the system as,

$$I_g = 2m(d_2^2 - d_1^2)T_0^2/(T_2^2 - T_1^2)\,\text{kg-m}^2.$$

A graph is plotted with length on the X-axis and T_0^2 on the Y-axis. The resulting graph will appear as follows.

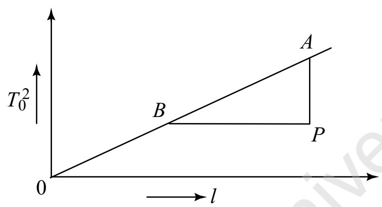


Fig. 1.13 Graph of a linear plot between T_0^2 and l

Table for calculating l/T_0^2

S.No	Length	Time period	$\frac{l}{T_0^2}$
1.			
2.			
3.			

The graph in Fig. 1.13 shows that as the length increases, the time period also increases which shows a linear plot. From this graph, a slope namely $\frac{l}{T_0^2}$, can be calculated and substituted in the second part of the experiment. The value of the moment of inertia that we have obtained from the first part of the experiment can be substituted in the formula, $\eta = \frac{8\pi I l}{T_0^2 R^4}$ (N/m^2), to determine the rigidity modulus of the given wire. The screw gauge is used in the second part of the experiment to find out the radius of the given wire by measuring its diameter (or thickness) in mm. The screw gauge readings are also recorded.

Zero Error Correction (ZEC)...... (mm) Least Count (L.C.) ....(mm)

Table for calculating the mean diameter of the wire

S. No	PSR(mm)	HSC	OR = PSR + HSC X L.C.	TR = OR	±ZEC
1.					
2.					
3.					
4.					
5.					
			Mean diameter = ... $\times 10^{-3}$ m		

The mean diameter and hence the mean radius R is calculated.

Note: PSR = Pitch scale reading
HSC = Head scale count (coincidence)
OR = Observation reading
TR = Total reading

Applications of Torsional Pendulum

1. The working of "Torsion pendulum clocks" (also called torsion clocks or pendulum clocks), is based on torsional oscillation.
2. The freely decaying oscillation of Torsion pendulum in medium (like polymers), helps to determine their characteristic properties.
3. Some researchers are trying to determine frictional forces between solid surfaces and flowing liquid environments using forced torsion pendulums.

1.8 BENDING OF BEAM

A rod or bar of uniform cross-section of a homogeneous, isotropic elastic material with a length that is much greater than its other dimensions is called a beam. Since the length of the beam is much greater than its other dimensions, shearing stresses can be neglected. When a beam is supported at one end and loaded at the other end, it is called a *cantilever*. When such a beam is fixed at one end and loaded at the other, within the limits of perfect elasticity, the loaded end sinks a little. The upper surface of the beam gets stretched and assumes a convex form and its lower surface gets compressed and assumes a concave form.

We will make the following assumptions while discussing the bending of beams:

1. Weight of the beam can be neglected in comparison to the load.
2. Shearing forces can be neglected.
3. There is no change in the cross section of the beam as it bends. This ensures that the geometric moment of inertia does not change as the beam bends.
4. The curvature of the bent beam is small.

1.9 BENDING MOMENT OF BEAMS

Let AB be a beam fixed at the end A and loaded at the end B, with EF as neutral axis. Let us consider the equilibrium of a section $PBCP'$ of it, cut by a plane PP' at right angles to its length and the plane of bending as shown in Fig. 1.14.

An equal and opposite reactional force W must be acting vertically upwards along PP', since the beam is fixed at the end AD.

The couple due to these two equal and opposite forces tends to rotate or bend the beam in the clockwise direction. This couple, due to the load applied to the free end of the beam is thus the "bending couple" and the moment of this couple is called the "bending moment", M.

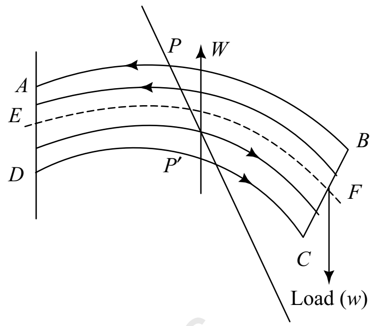


Fig. 1.14 Bending moment of beam

Since the beam is in equilibrium, there must obviously be an equal and opposite couple also acting on the beam.

Referring to section $PBCP'$, we know that the extended filaments above the neutral axis EF are in a state of tension and thus exert an "inward pull" on the filaments adjacent to them towards the fixed end of the beam. Similarly, the shortened filaments below EF are in a state of compression and exert an "outward push" on the filaments adjacent to the beam towards the loaded end of the beam.

The moment of these stresses thus balances the bending moment M due to the load when the beam is in equilibrium and must also be of the nature of the couple opposing or, resisting the bending couple due to the load.

The moment of this balancing couple, due to tensile and compressive stress in the upper and lower halves of the beam respectively, is referred to as "moment of the resistance to bending" in engineering practice and acts in the plane of the bending. Since it is equal in magnitude to the moment of the bending couple, it is also called *bending moment of the beam*.

Expression for the bending moment Let us consider a beam under the action of deforming forces. The beam bends into a circular arc. Figure 1.15 shows a portion of the beam.

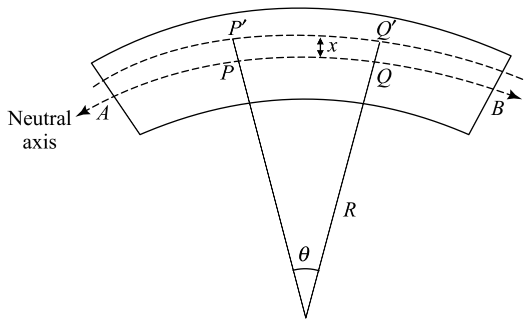


Fig. 1.15 Portion of the beam with radius of curvature R

Let AB be the neutral axis of the beam. Here the filaments above AB are elongated and the filaments below AB are compressed. The filament AB

remains unchanged. Let PQ be the arc chosen on the neutral axis. If R is the curvature of the neutral axis and θ is the angle subtended by it at its centre C. Thus, the length of arc $PQ = R\theta$ (1.77)

Consider a filament $P'Q'$ at a distance 'x' from the neutral axis.

$$\therefore \quad \text{The extended length} = P'Q' = (R + x)\theta \tag{1.78}$$

Here, increase in length = $P'Q' - PQ$

or, $(R + x)\theta - R\theta$

$$\therefore \quad \text{Increase in length} = x\theta \tag{1.79}$$

$$\text{Linear strain} = \frac{\text{Increase in length}}{\text{Original length}} = \frac{x\theta}{R\theta}$$

$$\therefore \quad \text{Linear strain} = \frac{x}{R} \tag{1.80}$$

$$\text{The Young's modulus of a material} = Y = \frac{\text{Stress}}{\text{Strain}}$$

$$\therefore \quad \text{Stress} = Y \times \text{strain} \tag{1.81}$$

Using Eq. (1.80), we get,

$$\text{stress} = Y \cdot \frac{x\theta}{R\theta} = \frac{Yx}{R}$$

If δA is the area of cross-section of the filament $P'Q'$. Then, the tensile force on the area (δA) = Stress × Area

$$\therefore \quad \text{Tensile force} = \frac{Yx}{R} \cdot (\delta A)$$

Moment of a force = Force × perpendicular distance

∴ Moment of a tensile force about the neutral axis (AB) or (PQ)

$$= \frac{Yx}{R} \cdot (\delta A)x$$

$$\therefore \quad PQ = \frac{Y}{R} \cdot (\delta A)x^2$$

The moment of force acting on both upper and lower halves of the neutral axis can be obtained by summing all the moments of tensile and compressive forces about the neutral axis.

$$\therefore \quad \text{the moment of all forces about the neutral axis} = \frac{Y}{R} \cdot \sum x^2(\delta A)$$

Here $I_g = \sum x^2(\delta A) = AK^2$ is called the geometrical moment of inertia, where A is the total area of the beam and K is the radius of gyration.

Hence, total moment of all the forces or internal bending moment

$$= \frac{YI_g}{R} \tag{1.82}$$

Special cases

(i) Rectangular cross-section

If 'b' is the breadth and 'd' is the thickness of the beam, then area $A = bd$

and $$K^2 = \frac{d^2}{12}$$

$$\therefore \quad I_g = AK^2 = \frac{bd \cdot d^2}{12} = \frac{bd^3}{12}$$

Substituting the value of I_g in Eq. (1.82), we get

Bending moment for a rectangular cross-section = $\frac{Ybl^3}{12R}$

(ii) Circular cross-section

For a circular cross-section if 'r' is the radius, then area $A = \pi r^2$ and

$$K^2 = \frac{r^2}{4}$$

$$\therefore \quad I_g = AK^2 = \frac{\pi r^2 \times r^2}{4} \quad \text{i.e., } I_g = \frac{\pi r^4}{4}$$

Using this in Eq. (1.82), we get,

Bending moment of a circular cross-section = $\frac{\pi Y r^4}{4R}$

1.10 CANTILEVER: THEORY AND EXPERIMENT

A cantilever is a beam fixed horizontally at one end and loaded at the other end.

If a load 'W' is applied at the free end, a couple is created between two forces i.e., (a) force (W) applied at the free end towards downward direction and (b) reaction (R) acting in the upward direction at the supporting end as shown in Fig. 1.16.

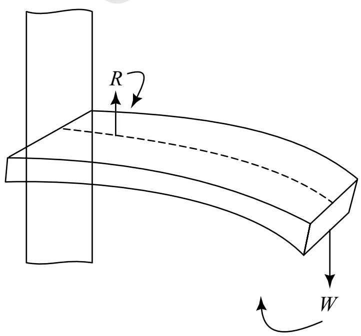


Fig. 1.16 Cantilever beam

Under equilibrium condition,

External bending moment = Internal bending moment

Experiment for calculating depression of a cantilever-loaded at one end Let 'l' be the length of the cantilever OA fixed at 'O'. Let 'W' be the weight suspended at the free end of the cantilever. Due to the load applied, the cantilever moves to a new position OA' as shown in Fig. 1.17.

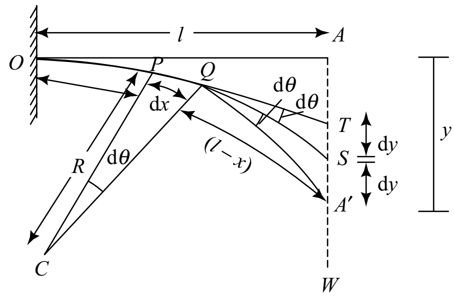


Fig. 1.17 Cantilever-loaded at its ends

Let us consider an element PQ of the beam of length dx, at a distance $OP = x$ from the fixed end. Let 'C' be the centre of curvature of the element PQ and let 'R' be the radius of curvature.

Due to the load applied at the free end of the cantilever an external couple is created between the load W at 'A' and the force of reaction 'Q'. Here the arm of the couple (distance between the two equal and opposite forces) is $l - x$.

$\therefore$ The external bending moment $= W(l - x)$ (1.83)

We know that the internal bending moment under equilibrium condition,

$$= \frac{YI_g}{R} \tag{1.84}$$

External bending moment = Internal bending moment

$\therefore$ Eq. (1.83) = Eq. (1.84)

i.e., $$W(l - x) = \frac{YI_g}{R}$$

or $$R = \frac{YI_g}{W(l - x)} \tag{1.85}$$

Two tangents are drawn at points P and Q, which meet the vertical line AA' at T and S respectively.

Let the smallest depression produced from T to $S = dy$

and let the angle between the two tangents $= d\theta$

Then we can write, the angle between CP and CQ is also $d\theta$.

i.e., $\angle PCQ = d\theta$.

Thus, the arc length $PQ = R.\, d\theta = dx$

or $$d\theta = \frac{dx}{R} \tag{1.86}$$

Substituting Eq. (1.85) in Eq. (1.86)

$$d\theta = \frac{dx}{[YI_g / W(l - x)]}$$

or $$d\theta = \frac{d\omega}{YI_g}\,(l - x)\,dx \tag{1.87}$$

From ΔQA′S we can write,

$$d\theta = \frac{dy}{(l - x)}$$

If $d\theta$ is very small, then we can write,

$$dy = d\theta\,(l - x) \tag{1.88}$$

Using Eqs (1.87) and (1.88), we have,

$$dy = \frac{W}{YI_g}\ (l\ - x)^2 . dx \tag{1.89}$$

Therefore, total depression at the free end of the cantilever can be derived by integrating Eq. (1.89) within the limits 0 to '*l*'.

$$\therefore \qquad y = \frac{W}{YI_g}\int_0^l (l - x)^2 . dx$$

$$= \frac{W}{YI_g}\int_0^l (l^2 - 2lx + x^2)dx$$

$$= \frac{W}{YI_g}\int_0^l \left(lx^2 - \frac{2lx^2}{2} + \frac{x^3}{3} \right) dx$$

$$\therefore \qquad y = \frac{W}{YI_g}\left[l^3 - l^3 + \frac{l^3}{3} \right]$$

$$\therefore \qquad y = \frac{W}{YI_g} \cdot \left(\frac{l^3}{3} \right)$$

∴ Depression of cantilever at free end '*y*' is,

$$y = \frac{Wl^3}{3YI_g} \tag{1.90}$$

Special cases

(i) Rectangular cross-section

If '*b*' is the breadth and '*d*' is the thickness of the beam then we know

$$I_g = \frac{bd^3}{12}$$

Substituting the value of I_g is Eq. (1.90), we can write, the depression produced at the free end for a rectangular cross-reaction

$$y = \frac{Wl^3}{3Y\left(\frac{bd^3}{12} \right)}$$

$$\therefore \qquad y = \frac{4Wl^3}{Ybd^3}$$

(ii) Circular cross-section

If '*r*' is the radius of circular cross section, then we know that,

$$I_g = \frac{\pi r^4}{4}$$

Substituting the value of I_g in Eq. (1.90), we can write

$$\text{Depression produced } y = \frac{Wl^3}{3Y(\pi r^4/4)}$$

$$y = \frac{4Wl^3}{3\pi r^4 Y}$$

Note: The angle between the tangents at the end of the cantilever can be obtained by integrating within the limits 0 to '*l*'.

$$\therefore \qquad \theta = \int_0^l \left(\frac{W}{Y} - xdx \text{ or } \theta = \frac{Wl^2}{2YI_g} \right)$$

Example 1.7 The free end of a given cantilever depresses 5 mm under a certain load. Calculate the depression produced if the length is increased to three times its original length.

Solution From Eq. (1.90), the depression y of the free end is given by

$$y = \frac{Wl^3}{3YI_G} \qquad (1.91)$$

If y_1, l_1 represent the original depression and length respectively and y_2, l_2 represent the corresponding final quantities, then we have,

$$\frac{y_2}{y_1} = \left(\frac{l_2}{l_1}\right)^3$$

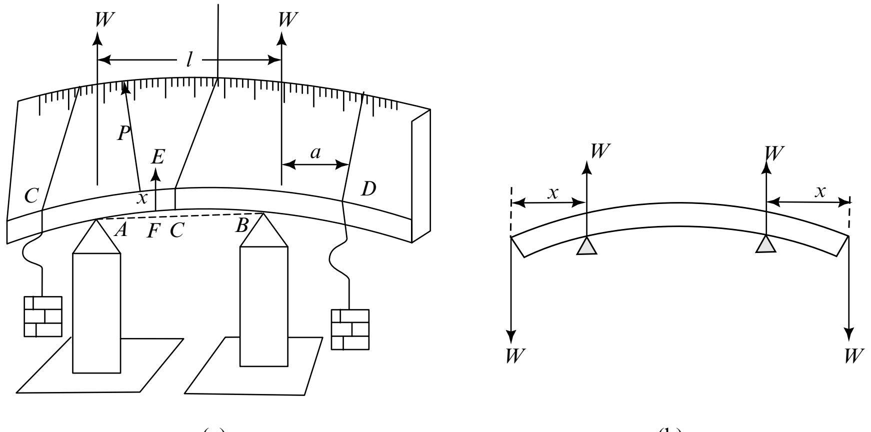
$$\text{or} \qquad y_2 = \left(\frac{l_2}{l_1}\right)^3 y_1$$

yielding, $y_2 = (3)^3 \times 5 \text{ mm} = 27 \times 5 = 135 \text{ mm}$

1.11 UNIFORM BENDING: THEORY AND EXPERIMENT

In uniform bending, there is an elevation at the centre of the beam due to the weights loaded at both ends.

Let us consider a beam of negligible mass, supported symmetrically on the two knife edges *A* and *B* as shown in Fig. 1.18. Let the length between *A* and *B* be '*l*'. Let equal weights *W*, be added to either ends of the beam *C* and *D* respectively.



(a) (b)

Fig. 1.18 Cantilever with uniform bending (a) experimental set-up (b) schematic layout

Let the distance CA = 3D = a

Due to the load applied the beam bends from position F to E as an arc of a circle and produces an elevation 'x' from position F to E. Let 'W' be the reaction produced at the points A and B which acts vertically, upwards as shown.

Consider a point 'P' on the cross-section of the beam. Then, the forces acting on the part PC of the beam are

1. Force W at 'C' and
2. Reaction W at 'A'

Let the distance $PC = a_1$ and $PA = a_2$ (as shown in Fig. 1.19), then the external bending moment about P, is

$$M_p = W \times a_1 - W \times a_2$$

Here, the clockwise moment is taken as negative and the anticlockwise moment is taken as positive.

External bending moment about P can be written as $M_p = W.(a_1 - a_2)$

$$M_p = W.a \tag{1.92}$$

We know that the internal bending moment

$$= \frac{YI_g}{R} \tag{1.93}$$

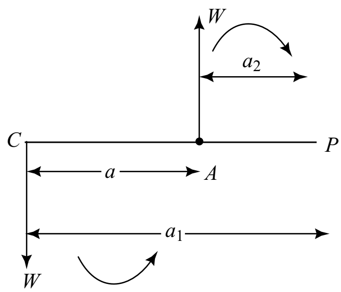


Fig. 1.19 Diagram for bending moment

Under equilibrium condition,

internal bending moment = external bending moment

Therefore, from Eq. (1.92) and Eq. (1.93), we get,

$$W.a = \frac{YI_g}{R} \tag{1.94}$$

Since for a given load, W, Y, I_g, a, R are constants, therefore, this type bending is called *uniform bending*. Here, it is found that the elevation 'x' forms an arc of the circle of radians 'R' as shown in Fig. 1.20.

In ΔAFO, we can write

$$OA^2 = AF^2 + FO^2$$

Since $OF = FE$

$$OA^2 = AF^2 + FE^2$$

Rearranging,

$$AF^2 = FE\left[\frac{OA^2}{FE} - FE\right] \qquad (1.95)$$

Here, $AF = \dfrac{l}{2}$, $FE = x = \dfrac{R}{2}$; $OA = R$

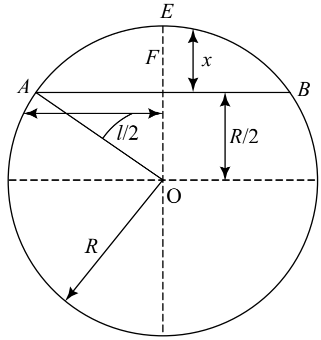


Fig. 1.20 Elevation of the central part

$\therefore$ Equation (1.95) can be written as

$$\left(\frac{l}{2}\right)^2 = x\left[\frac{R^2}{(R/2)} - x\right]$$

$$\frac{l^2}{4} = x[2R - x]$$

$$\frac{l^2}{4} = 2xR - x^2$$

If the elevation 'x' is very small, then the term x^2 can be neglected.

Therefore, we can write $\dfrac{l^2}{4} = 2xR$.

or $\qquad x = \dfrac{l^2}{8R}$

Radius of curvature $R = \dfrac{l^2}{8x}$ $\qquad$ (1.96)

Substituting for 'R' in Eq. (1.94)

we get, $\quad W.a = \dfrac{YI_g}{(l^2/8x)}$

or $\qquad W.a = \dfrac{8YI_g x}{l^2}$

$\therefore$ the elevation of point 'E' above A is

$$x = \frac{Wal^2}{8YI_g} \qquad (1.97)$$

1.12 NON-UNIFORM BENDING: THEORY AND EXPERIMENT

Theory In solids, Young's modulus is defined as the ratio of the longitudinal stress over longitudinal strain, in the range of elasticity the Hooke's law holds (stress is directly proportional to strain). It is a measure of stiffness of elastic material. Young's modulus of elasticity was discovered by Thomas Young, a 19th century British scientist. The ratio between the stress and the strain in a solid material is known as 'modulus of elasticity'. Stress can be of two types, namely linear stress and normal stress. Linear stress is applied to a material along its length and normal stress is applied in a direction perpendicular to the plane of a material. The SI unit of Young's modulus is Newton/metre^2 or Pascal.

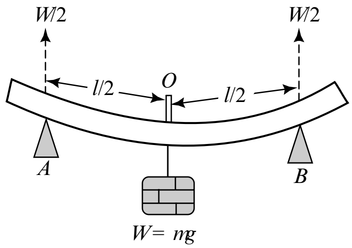


Fig. 1.21 Non-uniform bending method (Theory)

If a wire of length l and area of cross-section 'A' be stretched by a force F and if a change (increase) of length 'l' is produced, then,

$$Y = \frac{\text{Normal stress}}{\text{Linear strain}} = \frac{F/A}{\Delta l/l} = \text{Stress/Unit area (N/m}^2\text{)}.$$

Let AB be a beam that is supported on two knife edges. The length of the beam is l cm. The beam is now loaded at its middle point O by a weight as shown in Fig. 1.21,

$$W = mg \tag{1.98}$$

The reaction at the knife edges will be $W/2$ at A and at B respectively. The beam now bends because of the load mg. The beam may be regarded as two cantilevers whose free end carries a load of $W/2$ each of length $l/2$ and at a fixed point O. Using the theory of the cantilever, where the equilibrium bending moment is equal to the restoring moment we have,

$$\frac{Y}{R} I_g = W(l - x) \tag{1.99}$$

where, Y is the Young's modulus of elasticity, R is the reaction, I_g is the geometric moment of inertia, and $I_g = AK^2$, A being the total area of a section of the beam, and K being the radius of gyration of the beam. If y is the depression of the beam at O, the radius of curvature of the beam R is given by,

$$\frac{1}{R} = \mathrm{d}^2y/\mathrm{d}x^2 \ .1/\left[1 + \left(\frac{dy}{dx}\right)^2\right] \tag{1.100}$$

where, $\frac{dy}{dx}$ is the slope of the tangent at the point (x, y). Since the slope is small, $\left(\frac{dy}{dx}\right)^2$ is negligible and hence the expression becomes,

$$\frac{1}{R} = \mathrm{d}^2y/\,\mathrm{d}x^2 \tag{1.101}$$

Comparing Eq. (1.101) with Eq. (1.99), we get,

$$\mathrm{d}^2y/\,\mathrm{d}x^2 = W(l-x)/\ YI_g \tag{1.102}$$

On integrating Eq. (1.102), we get,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{W(lx - x^2/2)}{YI_g} + C_1 \tag{1.103}$$

where, C_1 is the constant of integration. When $x = 0$, that is, at the point A, the tangent is horizontal and hence the slope, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, so that on using this, we get $C_1 = 0$. Hence,

$$\frac{dy}{dx} = \frac{W(lx - x^2/2)}{YI_g} \tag{1.104}$$

Now, once again, on integration, we get,

$$y = \frac{W(lx^2/2 - x^3/6)}{YI_g} + C_2 \tag{1.405}$$

Applying the same condition, we find that at $x = 0$, $C_2 = 0$. Hence,

$$y = \frac{W(lx^2/2 - x^3/6)}{YI_g} \tag{1.106}$$

This relation gives the depression of a point B at a distance x from a fixed end. At the free end, $x = 1$, and therefore, the depression produced at the free end is

$$y_0 = \frac{W(l^3/2 - l^3/6)}{YI_g} = Wl^3/\ 3YI_g \tag{1.107}$$

For a beam of rectangular cross-section, we have, $I_\mathrm{g} = \frac{bd^3}{12}$ (1.108)

Hence, on substituting Eq. (1.108) in Eq. (1.107), we get,

$$Y = \frac{4Wl^3}{Ybd^3} \tag{1.109}$$

When $W = mg$ is substituted in Eq. (1.109), we get,

$$Y = \frac{4mgl^3}{Ybd^3}\ (\mathrm{N/m^2})$$

Thus, the Young's modulus of elasticity of a rectangular beam (wooden scale) is calculated using the non-uniform bending method.

Experiment The aim of the experiment is to find the Young's modulus of elasticity of a wooden metre scale, say, by the non-uniform bending method.

For this, the apparatus shown in Fig. 1.22 is used, and the beam is supported between two knife-edges AB. A pin is inverted and its tip faces the top at O. A Vernier microscope is kept to study the variation of the horizontal cross-wire(depression) of the beam (scale), each time, when a mass is either loaded or unloaded from the beam. The distance between the two-knife edges is measured as L (m).

The wooden scale is tied with an inverted pin at the middle of its length. A weight hanger containing weights of multiples of 50 gm each is suspended at the centre of the beam. A travelling microscope is taken and its horizontal cross-wire is adjusted to the tip of the beam by keeping a 50 gm mass in the beginning. The reading is noted. Then, another 50gm mass is added to this and again the depression of the pin is followed by the travelling microscope, and the reading is taken. The experiment is repeated for another two or three masses, each of 50 gm, is added to the existing mass. Readings are noted. Then the reverse procedure, namely every mass of 50 gm are unloaded, and each time the elevation of the pin is traced by the microscope, and then readings are taken. So, a set of readings for both the loading and unloading masses will now become available. The average is found for all these readings and the depression y is found by subtracting two successive readings, taken two at a time. In the second part of the experiment, a Vernier caliper is used to measure the breadth of the scale and a screw gauge is used to measure the thickness of the scale. The readings are recorded as shown in the following tables.

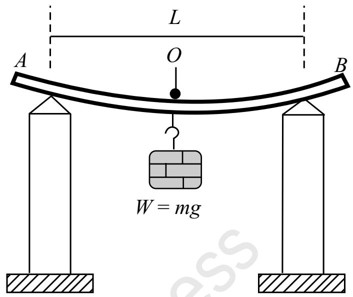


Fig. 1.22 Experimental set-up for non-uniform bending method

$$ZEC = Z.E. \times L.C. = \text{................} \text{ (cm)}$$

Table for Microscope Readings

S. No.	Length between the knife edges(l) m	Load (K gm)	Microscope readings Loading(P)	Unloading(Q)	Mean (P + Q)/2	Depression (y) for 50 gm
1		W				
2		W +50 gm				
3.		W +100 gm				
4.		W + 150 gm				
						Mean (y_1)
1.		W				
2.		W + 50 gm				
3.		W + 100 gm				

4.		W + 150 gm				
						Mean (y_2)
1.		W				
2.		W + 50 gm				
3.		W + 100 gm				
4.		W + 150 gm				
						Mean (y_3)

Table for calculating breadth of the scale using Vernier Caliper

ZEC = Z.E. × L.C. = ... (cm)

S. No	MSR (cm)	VSC	MSR + VSC × L.C.	CR	TR = CR ± ZEC

Mean breadth= (cm) =............... $\times 10^{-2}$ m.

Note: MSR = Main scale reading
VSC = Vernier scale coincidence
CR = Corrected reading
TR = Total reading

Using the formula $Y = \dfrac{4mgl^3}{Ybd^3}$ (N/m^2), we can calculate the Young's Modulus of elasticity, by theory. A graph is plotted between L^3 vs y (depression) as shown in Fig. 1.23. Knowing the three different lengths and three different mean depression values, we get a linear dependence plot as shown below.

Table for calculating thickness of the scale using Screw Gauge

ZEC = Z.E. × L.C. = ... (mm) L.C. = ... (mm)

Sl.No	PSR (mm)	HSC	HSC × L.C.	PSR + HSC × L.C.	CR	TR = CR ± ZEC(mm)
1.						
2.						
3.						
4.						
5.						

Mean thickness = ... (mm) = ... $\times 10^{-3}$ m.

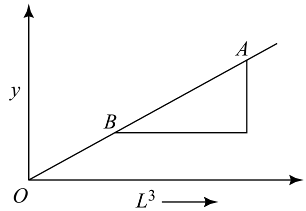


Fig. 1.23 Cube of length vs depression y in a non-uniform bending method

From the graph, the slope is found to be y/L^3. The reciprocal of the slope is then taken to be L^3/y and substituted in the above formula to get the experimental value of Y, the Young's modulus of elasticity of a given wooden beam (scale).

Example 1.8 Calculate the Young's modulus in the cantilever depression method used. The length of the cantilever is 1 m which is suspended with a load of 150 gm. The depression is found to be 4 cm. The thickness of the beam is 5 mm and breadth of the beam is 3 cm.

Solution Here, $Y = \dfrac{4mg\,l^3}{Ybd^3}$ (N/m^2).

Substituting for various terms, we get,

$4 \times 9.8 \times 1 \times 1 \times 1 \times 150 \times 10^{-3}/[3 \times 10^{-2} \times (5 \times 10^{-3})^3 \times 4 \times 10^{-2}]$

$= 5.88 / 1.5 \times 10^{10} = 3.92 \times 10^{10}$ N/m^2

Example 1.9 A circular and a square cantilever are made of a same material and have equal area of cross-section and length. Find the ratio of their depression for a given load.

Solution The geometric moment of inertia for a circular section $I_c = \pi r^4/4$. The geometric moment of inertia for a square section $I_s = \dfrac{bd^3}{12}$. (for a square $b = d = a$) Therefore,

$$I_s = a^4/4$$

For a circular cantilever, for a given load, $y_{\text{circular}} = Mgl^3/3YI_c$; and for a square $y_{\text{square}} = Mgl^3/3YI_s$. Hence, taking the ratio, we get,

$y_{\text{circular}}/y_{\text{square}} = I_s/I_c = [a^4/4]/[\pi r^4/4] = a^4/3\pi r^4$.

Also, the area of cross-sections are equal, i.e., $\pi r^2 = a^2$.

Hence, $y_c/y_s = \pi^2/3\pi = \pi/3$.

Example 1.10 A cantilever of rectangular cross-section has a length of 50 cm. Its breadth is 3 cm and its thickness is 0.6 cm. A weight of 1 Kg is attached at the free end. The depression produced is 4.2 cm. Calculate the Young's modulus of the material of the bar. Given that $g = 9.8$ m/sec^2.

Solution Using the formula, for the non-uniform bending method, i.e., $Y = \dfrac{4mg\,l^3}{Ybd^3}$ (N/m^2),

Substituting the given, we get, $Y = 1.8 \times 10^{10}$ N/m^2

Example 1.11 A uniform rectangular bar1m long, 2 cm broad, and 0.5 cm thick is supported on its flat face symmetrically on two knife edges 70 cm apart. If loads of

200 gm are hung from the two ends, the elevation of the centre of the bar is 48 mm. Find the Young's modulus of the bar.

Solution Applying the uniform bending method principle here, and using the formula $Y = \dfrac{4mgl^3}{Ybd^3}$, substituting the various terms, we obtain, $Y = 1.8 \times 10^{10}$ N/m^2

Example 1.12 A cantilever of length 0.5 m has a depression of 15 mm at its free end. Calculate the depression at a distance of 0.3 m from the fixed end.

Solution We know that the depression of the cantilever at a distance x from its fixed end is given by, $y = \dfrac{W}{YI}\left[\dfrac{lx^2}{2} - \dfrac{x^3}{6}\right]$, and at the free end, $\delta = \dfrac{Wl^3}{3YI}$

Here, l is the length of the cantilever,
W is the load,
Y is the Young's modulus of the material and
I is the geometric moment of inertia and = 0.015m.
On substituting these terms, we get, $y = 6.48$ mm $= 6.48 \times 10^{-3}$ m

1.13 I-SHAPED GIRDERS

The girders with upper and lower sections broadened and middle section tapered, so that it can withstand heavy loads over it is called as I-shaped girders (refer to Fig. 1.24). Since the girder looks like the letter I, they are known as I-shaped girders.

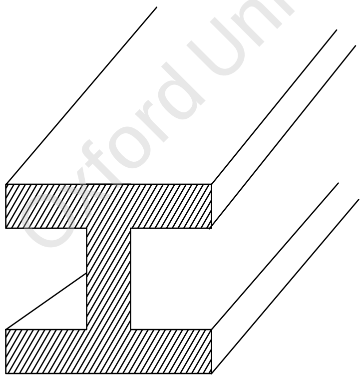


Fig. 1.24 I-shaped girders

Minimization of the depression produced

We know that the depression in the case of a rectangular section is given as,

$$y = \frac{4Wl^3}{Ybd^3}$$

Depression 'y' can be minimized by either decreasing the load (W) or the length of the girder (l) or by increasing the Young's modulus or the breadth (b) or the thickness (d) of the girder.

Since length 'l' is a fixed quantity, it cannot be decreased.

Therefore, breadth and thickness may be adjusted by increasing the depth and decreasing the breadth (since thickness increases by d^3). Thus volume of girder is increased and hence depression produced is reduced. Depression can also be reduced by properly choosing materials of high Young's modulus.

Applications of I-shaped Girders

1. They are used in the construction of bridges over rivers.
2. They are very much useful in the production of iron nails which are used in railway tracks.
3. They are used in supporting beams for ceilings in the construction of buildings.
4. They are used in construction of dams.

Advantages

1. More stability
2. More stronger
3. High durability

1.14 STRESS DUE TO BENDING OF BEAMS

Consider a beam under the action of a bending moment as shown in Fig. 1.25.

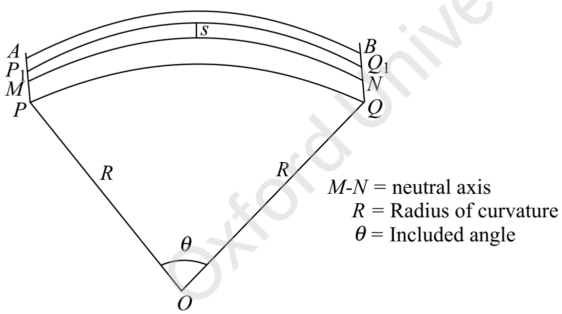


Fig. 1.25 Stress due to bending of a beam

Bending occurs with the centre of curvature at O, having a radius of curvature, included angle θ, and the neutral surface MN. If the longitudinal stress at a chosen filament P_1Q_1 at a distance s from the neutral surface MN is σ_s then the strain in P_1Q_1 is given by,

$$\text{Strain} = \frac{\text{Change in length}}{\text{Original length}}$$

$$= \frac{P_1Q_1 - PQ}{PQ} = \frac{(R+s)\theta - R\theta}{R\theta} = \frac{s\theta}{R\theta} = \frac{s}{R}$$

Also, $\quad \text{Strain} = \dfrac{\text{stress}}{\text{Young' smodulus}} = \dfrac{\sigma}{Y} = \dfrac{s}{R}$

gives us, $\sigma_s = \dfrac{s}{R}Y$

Here, $\frac{s}{R}$ is a constant for a particular cross-section of the beam. Hence, bending stress can be calculated at a particular cross-section and it is found that it is proportional to the distance from the neutral axis. For positive values of s, below the neutral axis, bending stress σ_s is positive or compressive stress and for negative values, above the neutral axis, the bending stress σ_s is taken as negative, or tensile stress (refer to Fig. 1.26).

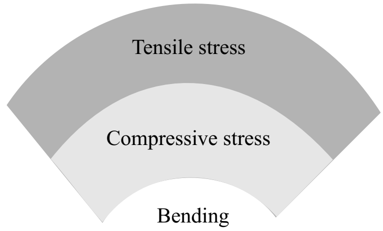


Fig. 1.26 Tensile stress and Compressive stress

IMPORTANT CONCEPTS

1. Stress is defined as the restoring force per unit area that brings back the body to its original state from the deformed state. Its unit is N/m^2.
2. Strain is defined as the change in dimension (fractional change) produced by an external force on the body. It has no unit.
3. There are 2 types of stress—(a) normal stress and (b) tangential stress or shearing stress.
4. There are 3 types of strain—(a) longitudinal or tensile strain, (b) shearing strain and (c) volumetric strain.
5. According to Hooke's law, stress is directly proportional to the strain produced, within the elastic limit.
6. There are 3 types of modulus of elasticity—(a) Young's modulus (Y), corresponding to longitudinal or tensile strain, (b) Bulk modulus (K) corresponding to volume strain, and (c) Rigidity modulus (η) or modulus corresponding to the shearing strain.
7. Poisson's ratio is defined as the ratio between lateral strain per unit stress(β) to the longitudinal strain per unit stress(α). Thus, $\sigma = \frac{\beta}{\alpha}$.
8. Relation between Y, η, and K is given as, $Y = \frac{9K\eta}{3K + \eta}$.
9. Factors affecting elasticity are stress, annealing, temperature, impurities, and nature of crystals.
10. The working stress on a material should be kept below the tensile stress in order to prevent the material from losing its elastic properties.
11. A shearing stress is equal to a linear tensile stress and to an equal linear compressive stress.
12. Torsion is the twisting of an object due to an applied torque.
13. Torsional stress is defined as a shearing stress produced when we apply the twisting moment to the end of a shaft about its axis.
14. Deformation means change in the shape or dimensions of a body as a result of stress and strain on a material.
15. When subjected to torsion, every cross-section of a circular shaft remains plane and undistorted. Cross-sections of non-circular (non-axisymmetric) shafts are distorted when subjected to torsion. Cross-sections for hollow and solid circular shafts remain plane and undistorted because a circular shaft is axisymmetric.

16. The moment of a force about a point is defined as the product of the magnitude of a force and the perpendicular distance from the point to the line of action of force.
17. Couple constitutes a pair of two equal and opposite forces acting on a body, in such a way that the lines of action of the two forces are not in the same straight line.
18. Torque is a rotating force and is equal to the moment of the couple. Torque is the product of one of the forces forming couple and the perpendicular distance between the two opposite forces.
19. A torsional pendulum consists of a rigid object (like a circular disc) suspended by a wire attached to a rigid support.
20. A beam is defined as a rod or a bar, either circular or rectangular, of uniform cross-section whose length is very much greater than its outer dimensions, such as breadth and thickness.
21. Total moment of all forces or the internal bending moment is = YI_g/R, where Y is Young's modulus of elasticity and $I_g = AK^2$, is the geometric moment of inertia. Here, A is the total area of the beam, K is the radius of gyration. R is the radius of curvature of the neutral axis.
22. A cantilever is a beam fixed horizontally at one end and loaded at the other end.
23. Expression for depression of a cantilever loaded at its free end: $y = Wl^3/3YI_g$, where Y is the Young's modulus of elasticity, y is the depression at the free end, $W = mg$ where m is the mass and g is the acceleration due to gravity, I_g is the geometric moment of inertia.
24. Uniform bending-elevation at the centre of the beam loaded at both ends: y= $3mgal^2/2bd^3 Y$, where y is the elevation at the centre, m is the mass, g is the acceleration due to gravity, a is the distance between any one of the knife-edges to the weight hanger, l is the distance between the knife edges, b is the breadth of the beam (scale), d is the thickness of the beam (scale), and Y is the Young's modulus of elasticity.
25. Non-uniform bending-expression for depression for a beam loaded at the centre: $Y = \dfrac{4mgl^3}{Ybd^3}$ (N/m^2), where Y is the Young's modulus of elasticity, m is the mass loaded in the centre each time, g is the acceleration due to gravity, l is the length between the knife edges, y is the depression at the centre, b is the breadth of the beam, and d is the thickness of the beam.
26. The girders with upper and lower sections broadened and the middle section tapered, so that it can withstand heavy loads over it is called I-section girder.
27. For a beam subjected to bending stress, it can be calculated at a particular cross-section and it is found that it is proportional to the distance from the neutral axis.

IMPORTANT FORMULAE

1. Deforming stress = $\dfrac{F}{A}$
2. Strain = $\dfrac{\text{Change in dimension}}{\text{original dimension}}$
3. Force (Hooke's law)

   $F = -k.\ X$
4. Tensile strength

   $= \dfrac{\text{Maximum Tensile load}}{\text{Original cross-sectional area}}$
5. Young's Modulus of elasticity

   $Y = \dfrac{FL}{Al}$

6. Rigidity Modulus: $\eta = \dfrac{Fl}{Al}$
7. Bulk modulus of elasticity: $K = \dfrac{pV}{\vartheta}$
8. Twisting couple per unit twist for a cylinder (or a wire): $C = \dfrac{\pi\eta\theta}{2l}r^4$
9. Twisting couple per unit twist for a hollow cylinder: $T = \dfrac{\pi\eta}{2l}\left(r_2^{\,4} - r_1^{\,4}\right)$
10. Period of the torsional pendulum: $T = 2\pi\sqrt{I/\kappa}$
11. Moment of inertia for a circular disc, suspended about a vertical axis ,using the torsional oscillation method:
$$I_g = 2m\left(d_2^2 - d_1^2\right)\left(\frac{T_0^2}{T_2^2 - T_1^2}\right)$$
12. Rigidity modulus of a wire using torsional pendulum method: $\eta = \dfrac{8\grave{A}Il}{T_0^2 R^4}$
13. For a cantilever, the depression due to a load at the free end: $y = \dfrac{Wl^3}{3YIg}$
14. The Young's modulus of a beam (wooden scale), using the non-uniform bending method:
$$Y = \frac{4mgl^3}{ybd^3}$$
15. The Young's modulus of a beam (wooden scale), using the uniform bending method:
$$Y = \frac{3mgal^2}{2bd^3y}$$

APPLICATIONS

1. Elastic constants, namely Young's modulus of elasticity, Rigidity modulus of elasticity, Bulk modulus of elasticity are all important properties of matter that deal with deformation and its consequences. These parameters have their applications in building a robust bridge, a high-rise building, or lay down concrete sleepers on a railway track, etc.
2. The I-shaped girders are another practical example of how a beam supports almost the entire weight of an object kept on it. This is used in engineering applications, such as construction of railway tracks, building bridges, constructing buildings, etc. Hence, the study of properties of matter is the first tool in engineering principles.

SELF-ASSESSMENT

Multiple-choice Questions

1.1 Young's modulus has units of
(a) N/m^3
(b) N/m^2
(c) $N\text{-}m^2$
(d) N/m

1.2 Correct expression for Bulk modulus is P
(a) $PV(\Delta V)$
(b) $\dfrac{P(\Delta V)}{V}$
(c) $\dfrac{PV}{(\Delta V)}$
(d) $\dfrac{PV^2}{(\Delta V)}$

1.3 The correct relation between Y and α is
(a) $Y = \alpha$
(b) $Y = \alpha^2$
(c) $Y = \dfrac{1}{\alpha^2}$
(d) $Y = \dfrac{1}{\alpha}$

1.4 Choose the correct expression,

(a) $K = \dfrac{1}{3(\alpha - 2\beta)}$

(b) $K = \dfrac{1}{3(\alpha - \beta)}$

(c) $K = \dfrac{3}{(\alpha - 2\beta)}$

(d) $K = \dfrac{3\alpha}{2\beta}$

1.5 For a cantilever loaded at the free end, the depression y at the free end is proportional to

(a) l^2

(b) l^3

(c) l

(d) $\frac{1}{2}I\omega^2$

1.6 Three wires of a same material are stretched by the same load. The dimensions are given below. Which one of them will elongate the most?

(a) diameter 1 mm, length 100 cm

(b) diameter 2 mm, length 200 cm

(c) diameter 0.5 mm, length 50 cm

(d) diameter 0.7 mm, length 150 cm

1.7 A wire of length L and radius r is fixed at one end and a force F is applied to the other end produces an extension l. The extension produced in another wire of the same material of length $2L$ and a radius $2r$ by a force $2F$ is

(a) l

b) $2l$

(c) $l/2$

(d) $4l$

1.8 The extension of a wire by the application of a load is 3 mm. The extension in a wire of the same material and length but half the radius by the same load will be

(a) 12.0 mm

(b) 0.75 mm

(c) 6.0 mm

(d) 1.5 mm

1.9 If the diameter of the suspension wire is doubled without changing the length in case of a torsional pendulum, the time period

(a) will increase

(b) will not be affected

(c) will decrease

(d) will double

1.10 A grandfather clock depends on the period of a pendulum to keep correct time. Suppose the clock is calibrated correctly and then a child slides the bob of the pendulum downward on the oscillating rod. Does the clock run?

(a) slow

(b) fast

(c) correctly

(d) will stop

1.11 Suppose the above grandfather clock is calibrated correctly at sea level and is taken to the top of a very tall mountain. Does the clock now run?

(a) slow

(b) fast

(c) correctly

(d) will stop

1.12 The total kinetic energy in pure rolling without sliding is:

(a) $\frac{1}{2}I\omega^2$

(b) $\frac{1}{2}mv^2$

(c) $\frac{1}{2}mv^2$

(d) $\frac{1}{2}I\omega$

1.13 The moment of inertia does not depend upon

(a) mass of the body

(b) the distribution of the mass of the body

(c) the angular velocity of the body and

(d) axis of rotation of the body

1.14 Without weighing, how will you distinguish between the two identical balls of same material but one of solid and the other one being hollow.
(a) by rolling them down an inclined plane in air
(b) by determining their moment of inertia about the centre
(c) by spinning them by equal torque
(d) all of the above

1.15 One solid sphere and a disc of same radius are falling along an inclined plane without slip. One reaches earlier than the other due to:
(a) different radius of gyration
(b) different size
(c) different friction
(d) different moment of inertia

1.16 A spiral spring is stretched by a weight attached to it. The strain will be
(a) elastic
(b) bulk
(c) shear
(d) tensile

1.17 The effect of temperature on the value of modulus of elasticity for various substances in general
(a) increases with increase in temperature
(b) remains constant
(c) decreases with rise in temperature
(d) none of the above

Review Questions

1.1 Define Young's modulus and give its unit.
1.2 What is compressibility?
1.3 Define modulus of rigidity.
1.4 What is the unit of Poisson's ratio?
1.5 Derive a relation between y and α.
1.6 Derive a relation between K, α and β.
1.7 Derive a relation between Y, k and β.
1.8 Derive a relation between η, α and β.
1.9 Derive a relation between Y, η and k.
1.10 Derive an expression for internal bending moment of a beam.
1.11 Define neutral axis.
1.12 How are the various filaments of a beam affected when the beam is loaded?
1.13 Mention the factors affecting elasticity of a material.
1.15 How does temperature and impurity in a material affect the elasticity of materials?
1.17 What are the types of stresses?
1.18 State Hooke's law.
1.19 What are the types of moduli of elasticity?
1.20 What is uniform bending?
1.23 Define elastic limit and plastic limit.
1.24 Define yield point.
1.25 What is Poisson's ratio?
1.26 Define torque.
1.27 Define shearing strain.
1.28 What is a stress–strain diagram? Write a short note on it.
1.29 Explain the factors affecting the elasticity of a material.

1.30 What is meant by bending of beams? Derive the expression for the bending moment for rectangular and circular cross-sections.
1.31 What is a cantilever? Derive the expression for the depression produced due to a load hanging at the end of a cantilever beam.
1.32 Describe how you will find elevation for a beam which is subjected to uniform bending.
1.33 Explain how you will find the Young's modulus of a material by using the non-uniform bending method.
1.34 Write a short note on I-shaped girders. Mention their uses.

Numerical Problems

1.1 A uniform rectangular bar 1 m long, 0.02 m broad, and 0.003 m thick is supported on two knife edges 0.7 m apart. When the loads of 0.2 Kg are hung from the ends, the elevation of the bar above its normal position is found to be 0.0022 m. Find the Young's modulus of the material of the bar.
1.2 In an experiment, a bar of length 1.5 m is clamped horizontally at one end and a load of 0.1 Kg is attached at its free end. Calculate the depression at the loaded end if $Y = 9.78 \times 10^{10}$ N/m^2 and the bar is of breadth 0.024 m and thickness is 0.005 m.
1.3 A uniform rectangular bar 1m long, 2 cm broad and 0.5 cm thick is supported on its flat face symmetrically on two knife edges 70 cm apart. If loads of 200 gm are hung from the two ends, the elevation of the centre of the bar is 48 mm. Find the Young's modulus of the bar.
1.4 A bar, one metre long with a square cross-section, of side 5 mm, is supported horizontally at its ends and is loaded at the middle point. It is depressed by 1.96 mm by a load of 100 gm, calculate the Young's modulus of the material of the bar.
1.5 A cantilever of steel fixed horizontally is subjected to a load of 225 gm at its free end. The geometric moment of inertia of the cantilever is 4.5×10^{-11} m^4. If the length of the cantilever and Young's modulus of steel are 1 m and 200×10^9 Pascal respectively, calculate the depression at the loaded end.
1.6 A circular and a square cantilever are made of same material and have an equal area of cross-section and length. Find the ratio of their depression for a given load.
1.7 A copper wire of 3 m length and 1mm diameter is subjected to a tension of 5 Kg weight. Calculate the elongation produced in the wire if the Young's modulus of elasticity of copper is 120 GP.
1.8 Determine Young's modulus of material of a rod, if it is bent uniformly over two knife-edges separated by a distance of 0.6 m and loads of 2.5 Kgs are hung at 0.18 m away from the knife edges. The breadth and thickness of the rod is 0.025 m and 0.005 m respectively. The elevation at the middle of the rod is 0.007 m.
1.9 Find the amount of work done in twisting a steel wire of radius 1mm and length 25 cm through an angle of 45°, the modulus of rigidity of steel being 8×10^{10} Nm^{-2}.
1.10 The end of a given strip, cantilever depresses 10 mm under a certain load. Calculate the depression under the same load for another cantilever of the same material, 2 times its length, 2 times in width, and three times its thickness (vertical).
1.11 A wire of length 1 m and diameter 1 mm is clamped at one of its ends. Calculate the couple required to twist the other end by 90°. Given that $\eta = 2.8 \times 10^{10}$ Nm^{-2}.