Applied Mathematics II

FEC 201 Applied Mathematics II

Based on the 2016–17 syllabus (CBCS) of University of Mumbai

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PREFACE

Mathematics as a subject is used in a wide range of fields. A thorough knowledge of the subject is the foundation on which engineers design solutions in all sectors of engineering, whether computer science, electrical, information technology, civil, or mechanical. Applied Mathematics deals with the application of mathematics to solve complex practical problems. It helps in explaining an observed scientific phenomena as well as in predicting new and/or overlooked phenomena. Applied Mathematics provides a balance between theory and practical aspects which helps engineers to overcome the challenges of modern engineering. Its deep understanding helps them to develop their technical skills which enable them to shape a technologically sound modern world.

ABOUT THE BOOK

Applied Mathematics II is specially designed for the first year engineering students of the University of Mumbai. The book covers all the topics taught in Applied Mathematics II course offered in the second semester and is written in a way to help students grasp the principles of important concepts clearly and easily. Each chapter starts with learning objectives that throw light on the important topics covered in the chapter. The book is written in a lucid manner where text is interspersed with necessary graphical representation of important topics, notes summarizing difficult and complicated topics, and step-wise solutions for problems.

With an aim to provide application-based learning, ample solved and unsolved problems have been provided. A list of formulae have been provided for easy reference and recapitulation. Each chapter ends with a bulleted summary to help students identify the important topics and study them thoroughly. We hope that our endeavour will enable them to gain the required knowledge of the subject and in the process move towards a better understanding of the field.

CONTENT AND COVERAGE

The book consists of 10 chapters and 2 model question papers. The model question papers have been included to help students understand the marking scheme and the exam pattern.

Chapter 1 provides general methods for solving exact differential equations, equations reducible to exact form, linear equations, and equations reducible to linear differential equations along with simple applications of ordinary differential equations to electrical and mechanical engineering problems.

Chapter 2 focuses on how to solve linear differential equations with constant coefficients of n^{th} order. It discusses the commonly employed technique for finding complementary functions and particular integral, and also explains the method for finding the solution of Cauchy's and Legendre's differential equations.

Chapter 3 discusses various numerical methods for finding the solution of ordinary differential equations.

Chapter 4 focuses on special class of functions called Gamma and Beta functions and their uses in integral calculus.

Chapter 5 explains the use of rule of differentiation under the integral sign in solving some special class of integrals.

Chapter 6 discusses numerical integration and methods for rectifying the curve when the equation of curve is given in Cartesian, polar, and parametric form.

Chapter 7 details the concept of evaluation of double integrals in Cartesian and polar co-ordinates.

Chapter 8 explains the concept of evaluation of triple integrals in Cartesian co-ordinate system, spherical polar co-ordinate system, and cylindrical polar co-ordinate system.

Chapter 9 focuses on simple applications of multiple integrals for finding area, mass, and volume.

Chapter 10 introduces the use of Scilab programming techniques for solving differential equations, tracing of curves, intersection of solids, etc.

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We feel great pleasure in presenting this book—Applied Mathematics II—for the first year engineering students of the University of Mumbai. Extensive teaching experience, interactions with the student community, and discussions with fellow colleagues motivated us to write this book. We would like to express our sincere gratitude to the management of Datta Meghe College of Engineering, Navi Mumbai, Smt. Indira Gandhi College of Engineering, Navi Mumbai, and A.C. Patil College of Engineering, Navi Mumbai. We are also grateful to Dr. S.D. Sawarkar, Principal, Datta Meghe College of Engineering, for his generous and invaluable support. We would also like to thank the following reviewers for their valuable feedback and their contribution in enhancing the content.

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We extend our deep sense of appreciation and indebtedness to our family members for being a driving force and inspiration for us. We would also like to thank the editorial team of Oxford University Press for helping us during this journey through their constructive approach.

We hope that the students and teachers of Mathematics will appreciate our efforts and receive this book enthusiastically. Any comments and suggestions for the improvement of the book are welcome and can be sent at atul.dubewar@gmail.com.

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DETAILED CONTENTS

CHAPTER 1

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

LEARNING OBJECTIVES

After reading this chapter, the readers will be able to

- identify the type of first-order and first-degree differential equations
- identify and solve exact differential equations
- identify and solve linear differential equations by the use of an integrating factor

1.1 INTRODUCTION

To represent mathematical models in the field of science and technology, we need first-order and first-degree differential equations. Differential equations in which first-order derivative occurs only once, whereas higher order derivatives do not occur, are called first-order and first-degree differential equations. The variable to be differentiated is called a dependent variable, and the variable which differentiates the dependent variable is called an independent variable.

If $f(x, y) = c$, where c is an arbitrary constant, is a function of x and y, then by total differentiation

$$\frac{\partial f(x, y)}{\partial x} dx + \frac{\partial f(x, y)}{\partial y} dy = 0$$

Let $M(x, y) = \dfrac{\partial f(x, y)}{\partial x}$ and $N(x, y) = \dfrac{\partial f(x, y)}{\partial y}$

$\therefore\ M(x, y)dx + N(x, y)dy = 0$

The above equation is a first-degree and first-order differential equation, and the function $f(x, y) = c$, where c is an arbitrary constant, is called a solution of the differential equation. As there is freedom to take any arbitrary constant c, every first-order, first-degree differential equation has infinite solutions.

So $f(x, y) = c$, where c is an arbitrary constant, is called a family of solutions.

In this module we are going to study the following types of first-order and first-degree differential equations:

1. Exact differential equations
2. Equations reducible to exact form
3. Linear differential equations of first order and first degree
4. Equations reducible to linear equation form

1.2 EXACT DIFFERENTIAL EQUATIONS

Definition: A first-order differential equation of the form

$$Mdx + Ndy = 0 \tag{1.1}$$

where M and N are functions of x and y, is said to be exact if the left-hand side is the total or exact differential of some function $u(x, y)$, i.e.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{1.2}$$

Then the differential equation (1.1) can be written as $du = 0$.
By integrating, we get

$$u(x, y) = c \tag{1.3}$$

Comparing Eqs (1.1) and (1.2), we get

$$\text{(a)} \quad \frac{\partial u}{\partial x} = M \qquad \text{(b)} \quad \frac{\partial u}{\partial y} = N \tag{1.4}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

But $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$ (Assumption)

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This condition is not only necessary but sufficient for $Mdx + Ndy = 0$ to be exact.

From Eq. 1.4(a), on integration w.r.t. x, we get

$$u = \int M\,dx + k(y) \tag{1.5}$$

where y is to be regarded as a constant, and $k(y)$ plays the role of a constant of integration. To determine the constant we derive $\dfrac{\partial u}{\partial y}$ from Eq. (1.5), and use Eq. 1.4(b) to get $\dfrac{dk}{dy}$, and then integrate.

Instead of Eq. 1.4(a), we may use Eq. 1.4(b),

$$u = \int N\,dy + l(x) \tag{1.6}$$

To determine $l(x)$ we derive $\dfrac{\partial u}{\partial x}$ from Eq. (1.6), and use Eq. 1.4(a) to get $\dfrac{dl}{dx}$, and then integrate.

Steps to Solve Exact Differential Equations

Check for the condition $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ to verify the exactness of the given equation.

Rule I:

(i) Integrate M with respect to x keeping y constant.
(ii) Integrate with respect to y, only those terms of N which do not contain x.
(iii) The final solution is of the form

$$\underset{\text{'}y\text{' constant}}{\int Mdx} \quad + \quad \underset{\text{Terms in '}N\text{' free from }x}{\int Ndy} = c$$

Rule II:
(i) Integrate N with respect to y, keeping x constant.
(ii) Integrate with respect to x, only those terms in M which do not contain y.
(iii) The final solution is of the form

$$\underset{\text{Terms in 'M' free from } y}{\int M dx} \quad + \quad \underset{\text{'x' constant}}{\int N dy = c}$$

EXAMPLES

Example 1.1 Solve $(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$.

Solution

This equation is of the form

$M\,dx + N\,dy = 0$

Here $M = (e^y + 1)\cos x$ and $N = e^y \sin x$

$$\frac{\partial M}{\partial y} = e^y \cos x \quad \text{and} \quad \frac{\partial N}{\partial x} = e^y \cos x$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given differential equation is exact and its solution is

$$\underset{\text{'y' constant}}{\int M\,dx} + \int [\text{Terms in } N, \text{which are free from } x]\,dy = c$$

$$\int \left[\left(e^y + 1\right)\cos x\right]dx + \int 0\,dy = c$$

$\therefore$ The solution is $\left(e^y + 1\right)\sin x = c$

Example 1.2 Solve $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0.$ **[MU 2000,15]** **[4 Marks]**

Solution

This equation is of the form

$M\,dx + N\,dy = 0$

Here $M = 1 + e^{\frac{x}{y}}$ and $N = e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right) \quad \text{and} \quad \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(-\frac{1}{y}\right) + \left(1 - \frac{x}{y}\right)e^{\frac{x}{y}}\frac{1}{y}$$

$$\frac{\partial M}{\partial y} = -\frac{x}{y^2}e^{\frac{x}{y}} \quad \text{and} \quad \frac{\partial N}{\partial x} = -\frac{e^{\frac{x}{y}}}{y} + \frac{1}{y}e^{\frac{x}{y}} - \frac{x}{y^2}e^{\frac{x}{y}} = -\frac{x}{y^2}e^{\frac{x}{y}}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given differential equation is exact and its solution is

$$\underset{\text{'y' constant}}{\int M\,dx} + \int [\text{Terms in } N, \text{which are free from } x]\,dy = c$$

$$\int\left(1+e^{\frac{x}{y}}\right)dx+\int 0\,dy=c$$

$$x+\frac{e^{\frac{x}{y}}}{\frac{1}{y}}=c$$

$\therefore$ The solution is $x+ye^{\frac{x}{y}}=c$

Example 1.3 Solve $\left(x^2-x\tan^2 y+\sec^2 y\right)dy=\left(\tan y-2xy-y\right)dx.$ **[MU 2001]**

Solution **[4 Marks]**

We have $\left(\tan y-2xy-y\right)dx-\left(x^2-x\tan^2 y+\sec^2 y\right)dy=0$

This equation is of the form

$$M\,dx+N\,dy=0$$

Here $M=\tan y-2xy-y$ and $N=-x^2+x\tan^2 y-\sec^2 y$

$$\frac{\partial M}{\partial y}=\sec^2 y-2x-1 \quad\text{and}\quad \frac{\partial N}{\partial x}=-2x+\tan^2 y$$

$$=\tan^2 y-2x$$

$$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

The given differential equation is exact and its solution is

$$\int\limits_{\text{'}y\text{' constant}} M\,dx+\int\left[\text{Terms in } N\text{, which are free from } x\right]dy=c$$

$$\int\left(\tan y-2xy-y\right)dx+\int -\sec^2 y\,dy=c$$

$$x\tan y-2y\frac{x^2}{2}-xy-\tan y=c$$

$$x\tan y-x^2y-xy-\tan y=c$$

$\therefore$ The solution is $\left(x-1\right)\tan y-xy\left(1+x\right)=c$

Example 1.4 Solve $\dfrac{dy}{dx}=\dfrac{y+1}{\left(y+2\right)e^y-x}.$ **[MU 2010]**

Solution **[4 Marks]**

We have $\dfrac{dy}{dx}=\dfrac{y+1}{\left(y+2\right)e^y-x}$

$$\therefore \frac{dy}{dx}\left[\left(y+2\right)e^y-x\right]=y+1$$

$$\left(y+1\right)dx-\left[\left(y+2\right)e^y-x\right]dy=0$$

This equation is of the form

$$M\,dx+N\,dy=0$$

Here $M=y+1$ and $N=-\left(y+2\right)e^y+x$

$$\frac{\partial M}{\partial y}=1 \qquad \text{and} \qquad \frac{\partial N}{\partial x}=1$$

$$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

The given differential equation is exact and its solution is

$$\int\limits_{\text{'}y\text{' constant}} M\,dx+\int[\text{Terms in } N\text{, which are free from } x]dy=c$$

$$\int(y+1)dx+\int-(y+2)e^y dy=c$$

$$x(y+1)-\int(y+2)e^y\,dy=c$$

$$x(y+1)-\left[(y+2)e^y-(1)e^y\right]=c \quad [\text{Integrating by parts}]$$

$$x(y+1)-(y+1)e^y=c$$

$\therefore$ The solution is $(y+1)(x-e^y)=c$

Example 1.5 Solve $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+(x+\log x-x\sin y)dy=0.$ **[MU 2006, 15]**

Solution **[4 Marks]**

This equation is of the form

$$M\,dx+N\,dy=0$$

Here $M=y\left(1+\frac{1}{x}\right)+\cos y$ and $N=x+\log x-x\sin y$

$$\frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y \qquad \text{and} \qquad \frac{\partial N}{\partial x}=1+\frac{1}{x}-\sin y$$

$$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

The given differential equation is exact and its solution is

$$\int\limits_{\text{'}y\text{' constant}} M\,dx+\int[\text{Terms in } N\text{, which are free from } x]dy=c$$

$$\int\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\int 0\,dy=c$$

$\therefore$ The solution is $y[x+\log x]+x\cos y=c$

Example 1.6 Solve $\left[y\sin(xy)+xy^2\cos(xy)\right]dx+\left[x\sin(xy)+x^2y\cos(xy)\right]dy=0.$ **[MU 1999, 2003]**

Solution **[4 Marks]**

This equation is of the form

$$M\,dx+N\,dy=0$$

Here $M=y\sin(xy)+xy^2\cos(xy)$

$$\frac{\partial M}{\partial y}=xy\cos(xy)+\sin(xy)+x\left[y^2(-\sin xy)x+\cos(xy)2y\right]$$

$$=xy\cos(xy)+\sin(xy)-x^2y^2\sin(xy)+2xy\cos(xy)$$

$$=3xy\cos(xy)+(1-x^2y^2)\sin(xy)$$

$$N = x\sin(xy) + x^2 y\cos(xy)$$
$$\frac{\partial N}{\partial x} = yx\cos(xy) + \sin(xy) + y\left[x^2(-\sin xy)y + \cos(xy)2x\right]$$
$$= xy\cos(xy) + \sin(xy) - x^2y^2\sin(xy) + 2xy\cos(xy)$$
$$= 3xy\cos(xy) + \left(1 - x^2y^2\right)\sin(xy)$$
$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given differential equation is exact and its solution is

$$\underset{\text{'}y\text{' constant}}{\int M\,dx} + \int[\text{Terms in } N, \text{ which are free from } x]\,dy = c$$

$$\int\left[y\sin(xy) + xy^2\cos(xy)\right]dx + \int 0\,dy = c$$
$$\frac{-y\cos(xy)}{y} + y^2\left[\frac{x}{y}(\sin xy) - (1)\frac{(-\cos xy)}{y^2}\right] = c$$
$$-\cos(xy) + xy\sin(xy) + \cos(xy) = c$$

$\therefore$ The solution is $xy\sin(xy) = c$

Example 1.7 Solve $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$ **[MU 2003, 12]**

Solution **[4 Marks]**

We have $\dfrac{dy}{dx} = -\dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x}$

$\therefore\ (y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$

This equation is of the form

$$M\,dx + N\,dy = 0$$

Here $M = y\cos x + \sin y + y$ and $N = \sin x + x\cos y + x$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1 \quad \text{and} \quad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$
$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given differential equation is exact and its solution is

$$\underset{\text{'}y\text{' constant}}{\int M\,dx} + \int[\text{Terms in } N, \text{ which are free from } x]\,dy = c$$

$$\int(y\cos x + \sin y + y)dx + \int 0\,dy = c$$
$$y\sin x + x(\sin y + y) = c$$

$\therefore$ The solution is $y\sin x + x\sin y + xy = c$

Example 1.8 Solve $\left[1 + \log(xy)\right]dx + \left[1 + \dfrac{x}{y}\right]dy = 0.$

Solution

This equation is of the form

$$M\,dx + N\,dy = 0$$

Here $M = 1 + \log(xy)$ and $N = 1 + \dfrac{x}{y}$

$$\frac{\partial M}{\partial y}=\frac{1}{xy}x=\frac{1}{y} \qquad \text{and} \qquad \frac{\partial N}{\partial x}=\frac{1}{y}$$

$$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

The given differential equation is exact and its solution is

$$\underset{'y'\text{ constant}}{\int M\,dx}+\int[\text{Terms in } N\text{, which are free from } x]dy=c$$

$$\int[1+\log(xy)]dx+\int 1\,dy=c$$

$$x+\int \log(xy)\cdot 1\,dx+y=c$$

$$x+\left[\log(xy)x-\int\frac{1}{xy}yx\,dx\right]+y=c$$

$$x+x\log(xy)-x+y=c$$

$\therefore$ The solution is $y+x\log(xy)=c$

Example 1.9 Solve $\left(x^2-4xy-2y^2\right)dx+\left(y^2-4xy-2x^2\right)dy=0.$ **[MU 2014]**

Solution **[4 Marks]**

This equation is of the form

$$Mdx+Ndy=0$$

Here $M=x^2-4xy-2y^2$ and $N=y^2-4xy-2x^2$

$$\frac{\partial M}{\partial y}=-4x-4y \qquad \text{and} \qquad \frac{\partial N}{\partial x}=-4y-4x$$

$$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

The given differential equation is exact and its solution is

$$\underset{'y'\text{ constant}}{\int M\,dx}+\int[\text{Terms in } N\text{, which are free from } x]dy=c$$

$$\int\left(x^2-4xy-2y^2\right)dx+\int y^2dy=c$$

$\therefore$ The solution is $\dfrac{x^3}{3}-2x^2y-2y^2x+\dfrac{y^3}{3}=c$

Example 1.10 Solve $\left(x\sqrt{x^2+y^2}-y\right)dx+\left(y\sqrt{x^2+y^2}-x\right)dy=0.$ **[MU 2014]**

Solution **[4 Marks]**

This equation is of the form

$$M\,dx+N\,dy=0$$

Here $M=x\sqrt{x^2+y^2}-y$ and $N=y\sqrt{x^2+y^2}-x$

$$\frac{\partial M}{\partial y}=\frac{xy}{\sqrt{x^2+y^2}}-1 \qquad \text{and} \qquad \frac{\partial N}{\partial x}=\frac{xy}{\sqrt{x^2+y^2}}-1$$

$$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

The given differential equation is exact and its solution is

$$\underset{\text{'}y\text{' constant}}{\int M\,dx} + \int [\text{Terms in } N, \text{ which are free from } x]\,dy = c$$

$$\int \left(x\sqrt{x^2+y^2} - y\right)dx + \int 0\,dy = c$$

$$\int x\sqrt{x^2+y^2}\,dx - \int y\,dx = c$$

$\therefore$ The solution is $\dfrac{\left(x^2+y^2\right)^{3/2}}{3} - yx = c$

EXERCISES

Solve the following differential equations:

1.1 $\dfrac{dy}{dx} = \dfrac{y}{2y\log y + y - x}$

[**Ans :** $y^2\log y - xy = c$]

1.2 $\left(y^2e^{xy^2} + 4x^3\right)dx + \left(2xye^{xy^2} - 3y^2\right)dy = 0$

[**Ans :** $e^{xy^2} + x^4 - y^3 = c$] **[MU 1999]**

1.3 $x\,dx + y\,dy = \dfrac{a(x\,dy - y\,dx)}{x^2+y^2}$

[**Ans :** $x^2 + y^2 + 2a\tan^{-1}\left(\dfrac{x}{y}\right) = c$] **[MU 1999]**

1.4 $4(x - 2e^y)dy + (y + x\sin x)dx = 0$

[**Ans :** $xy - x\cos x + \sin x - 2e^y = c$] **[MU 2013]**

1.3 REDUCTION OF NON-EXACT DIFFERENTIAL EQUATIONS TO EXACT DIFFERENTIAL EQUATIONS

Sometimes the given differential equation $Mdx + Ndy = 0$ is not exact i.e. $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, but it becomes exact by multiplication of a suitable factor called the *integrating factor*.

Standard Rules for Finding Integrating Factors

Rule I: If the given differential equation $Mdx + Ndy = 0$ is not exact and if $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ is a function of x only, say $f(x)$.

Then the integrating factor is $\text{I.F.} = e^{\int f(x)dx}$

The differential equation is then reduced to exact differential equation as

$$\left[(\text{I.F.})M\right]dx + \left[(\text{I.F.})N\right]dy = 0$$

Its solution is given as

$$\underset{\text{'}y\text{' constant}}{\int \left[(\text{I.F.})M\right]dx} + \int \left[\text{Terms in } (\text{I.F.})N, \text{ which are free from } x\right]dy = c$$

EXAMPLES

Example 1.11 Solve $\left(4xy+3y^2-x\right)dx+x\left(x+2y\right)dy=0.$ **[MU 2006, 12]**

Solution **[6 Marks]**

This equation is of the form

$$M\,dx+N\,dy=0$$

Here $M=4xy+3y^2-x$ and $N=x\left(x+2y\right)$

$$\frac{\partial M}{\partial y}=4x+6y \quad \text{and} \quad \frac{\partial N}{\partial x}=2x+2y$$

$$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

It is a non-exact differential equation. To find the integrating factor, we have

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\frac{2x+4y}{x\left(x+2y\right)}$$

$$=\frac{2}{x}\left(\frac{x+2y}{x+2y}\right)=\frac{2}{x}=f\left(x\right) \qquad \text{(say)}$$

$$\therefore \text{ I.F.}=e^{\int f(x)dx}=e^{\int\frac{2}{x}dx}$$

$$=e^{2\log x}=e^{\log x^2}=x^2$$

Now multiplying the given equation by I.F., we get

$$x^2\left[4xy+3y^2-x\right]dx+x^2\left[x\left(x+2y\right)\right]dy=0, \text{ which is exact}$$

Here $M'=x^2\left[4xy+3y^2-x\right]$ and $N'=x^3\left(x+2y\right)$

Its solution is given as

$$\int\limits_{\text{'}y\text{' constant}} M'dx+\int\left[\text{Terms in } N'\text{, which are free from } x\right]dy=c$$

$$\int x^2\left[4xy+3y^2-x\right]dx+\int 0dy=c$$

$$y\frac{4x^4}{4}+y^2\frac{3x^3}{3}-\frac{x^4}{4}=c$$

$\therefore$ The solution is $x^4y+x^3y^2-\dfrac{x^4}{4}=c$

Example 1.12 Solve $\left(x^4e^x-2mxy^2\right)dx+2mx^2y\,dy=0.$ **[MU 2003]**

Solution **[6 Marks]**

This equation is of the form

$$M\,dx+N\,dy=0$$

Here $M=x^4e^x-2mxy^2$ and $N=2mx^2y$

$$\frac{\partial M}{\partial y}=-4mxy \quad \text{and} \quad \frac{\partial N}{\partial x}=4mxy$$

$$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

It is a non-exact differential equation. To find the integrating factor, we have

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{-8mxy}{2mx^2y}=-\frac{4}{x}=f(x) \qquad \text{(say)}$$

$$\therefore \text{I.F.}=e^{\int f(x)dx}=e^{\int -\frac{4}{x}dx}$$

$$=e^{-4\log x}=e^{\log x^{-4}}=x^{-4}=\frac{1}{x^4}$$

Now multiplying the given equation by I.F., we get

$$\frac{1}{x^4}\left(x^4e^x-2mxy^2\right)dx+\frac{1}{x^4}\left(2mx^2y\right)dy=0, \text{ which is exact}$$

$$\left(e^x-\frac{2my^2}{x^3}\right)dx+\frac{2my}{x^2}dy=0$$

Here $M'=e^x-\dfrac{2my^2}{x^3}$ and $N'=\dfrac{2my}{x^2}$

Its solution is given as

$$\int\limits_{\text{'}y\text{' constant}} M'dx+\int[\text{Terms in } N', \text{ which are free from } x]\,dy=c$$

$$\int\left(e^x-\frac{2my^2}{x^3}\right)dx+\int 0\,dy=c$$

$$e^x-2my^2\left(\frac{x^{-2}}{-2}\right)=c$$

$\therefore$ The solution is $e^x+\dfrac{my^2}{x^2}=c$

Example 1.13 Solve $\left(2x\log x-xy\right)dy+2y\,dx=0.$ **[MU 2003]**

Solution **[6 Marks]**

This equation is of the form

$$M\,dx+N\,dy=0$$

Here $M=2y$ and $N=2x\log x-xy$

$$\frac{\partial M}{\partial y}=2 \quad \text{and} \quad \frac{\partial N}{\partial x}=2\left[x\times\frac{1}{x}+\log x\right]-y$$

$$=2(1+\log x)-y=2+2\log x-y$$

$$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

It is a non-exact differential equation. To find the integrating factor, we have

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2-2-2\log x+y}{2x\log x-xy}$$

$$=\frac{y-2\log x}{x(2\log x-y)}=-\frac{1}{x}=f(x) \qquad \text{(say)}$$

$$\therefore \text{I.F.}=e^{\int -\frac{1}{x}dx}=e^{-\log x}$$

$$= e^{\log x^{-1}} = \frac{1}{x}$$

Now multiplying the given equation by I.F., we get

$$\frac{1}{x}(2x\log x - xy)dy + \frac{1}{x}(2y)\,dx = 0, \text{ which is exact}$$

$$(2\log x - y)dy + \frac{2y}{x}\,dx = 0$$

Here $N' = 2\log x - y$ and $M' = \frac{2y}{x}$

Its solution is given as

$$\underset{\text{'}y\text{' constant}}{\int M'dx} + \int[\text{Terms in } N', \text{which are free from } x]\,dy = c$$

$$\int\left(\frac{2y}{x}\right)dx + \int -y\,dy = c$$

$\therefore$ The solution is $2y\log x - \frac{y^2}{2} = c$

Example 1.14 Solve $\left(y + \frac{y^3}{3} + \frac{1}{2}x^2\right)dx + \frac{1}{4}\left(x + xy^2\right)dy = 0.$ **[MU 2000, 08]**

Solution **[6 Marks]**

This equation is of the form

$$M\,dx + N\,dy = 0$$

Here $M = y + \frac{y^3}{3} + \frac{x^2}{2}$ and $N = \frac{1}{4}\left(x + xy^2\right)$

$$\frac{\partial M}{\partial y} = 1 + \frac{3y^2}{3} = 1 + y^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{1}{4}\left(1 + y^2\right)$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

It is a non-exact differential equation. To find the integrating factor, we have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1 + y^2 - \frac{1}{4}\left(1 + y^2\right)}{\frac{1}{4}\left(x + xy^2\right)}$$

$$= \frac{3}{4}\left(1 + y^2\right)\frac{1}{\frac{x}{4}\left(1 + y^2\right)} = \frac{3}{x} = f(x) \qquad \text{(say)}$$

$$\therefore \text{I.F.} = e^{\int f(x)dx} = e^{\int \frac{3}{x}dx}$$

$$= e^{3\log x} = e^{\log x^3} = x^3$$

Now multiplying the given equation by I.F., we get

$$x^3\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)dx + \frac{x^3}{4}\left(x + xy^2\right)dy = 0, \text{ which is exact}$$

Here $M' = x^3\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)$ and $N' = \frac{x^3}{4}\left(x + xy^2\right)$

Its solution is given as

$$\underset{\text{'}y\text{' constant}}{\int M'dx} + \int [\text{Terms in } N'\text{, which are free from } x]\, dy = c$$

$$\int x^3\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx+\int 0\,dy=c$$

$$y\frac{x^4}{4}+\frac{y^3}{3}\frac{x^4}{4}+\frac{x^6}{12}=c$$

$\therefore$ The solution is $3x^4y+x^4y^3+x^6=c$

Example 1.15 Solve $x\sin x dy+(xy\cos x-y\sin x-2)dx=0.$ **[MU 2015]**

Solution **[6 Marks]**

This equation is of the form

$$M\,dx+N\,dy=0$$

Here $M=xy\cos x-y\sin x-2$ and $N=x\sin x$

$$\frac{\partial M}{\partial y}=x\cos x-\sin x \quad \text{and} \quad \frac{\partial N}{\partial x}=x\cos x+\sin x$$

$$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

It is a non-exact differential equation. To find the integrating factor, we have

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\frac{x\cos x-\sin x-x\cos x-\sin x}{x\sin x}$$

$$=\frac{-2\sin x}{x\sin x}=-\frac{2}{x}=f(x) \quad (\text{say})$$

$$\therefore \text{I.F.}=e^{\int f(x)dx}=e^{\int -\frac{2}{x}dx}$$

$$=e^{-2\log x}=e^{\log x^{-2}}=x^{-2}$$

Now multiplying the given equation by I.F., we get

$$x^{-2}[xy\cos x-y\sin x-2]dx+x^{-2}[x\sin x]dy=0, \text{ which is exact}$$

$$\left[\frac{y\cos x}{x}-\frac{y\sin x}{x^2}-\frac{2}{x^2}\right]dx+\left[\frac{\sin x}{x}\right]dy=0$$

Here $M'=\dfrac{y\cos x}{x}-\dfrac{y\sin x}{x^2}-\dfrac{2}{x^2}$ and $N'=\dfrac{\sin x}{x}$

Its solution is given as

$$\underset{\text{'}y\text{' constant}}{\int M'dx} + \int [\text{Terms in } N'\text{, which are free from } x]\, dy = c$$

$$\int\left(\frac{y\cos x}{x}-\frac{y\sin x}{x^2}-\frac{2}{x^2}\right)dx+\int 0\,dy=c$$

$$y\int\frac{\cos x}{x}dx-y\int\frac{\sin x}{x^2}dx-\int\frac{2}{x^2}dx=c$$

Since $\int M'dx$ is difficult, we apply Rule II

$\therefore \int N'dy + \int$[Terms in M' which are free from y]$dy = c$

$$-\int \frac{2}{x^2}dx + \int \frac{\sin x}{x}dy = c$$

$$\frac{2}{x} + \frac{y\sin x}{x} = c$$

$\therefore$ The solution is $\dfrac{2}{x} + \dfrac{y\sin x}{x} = c$

EXERCISES

Solve the following equations:

1.5 $\left(x^2 + y^2 + 2x\right)dx + 2y\,dy = 0$

[**Ans :** $e^x(x^2+y^2) = c$]

1.6 $\left(x^3e^x - my^2\right)dx + mxy\,dy = 0$

[**Ans :** $2x^2e^x + my^2 = cx^2$]

1.7 $\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2y\,dy = 0$

[**Ans :** $\dfrac{1}{3}e^{1/x^3} - \dfrac{y^2}{2x^2} = c$] **[MU 2004, 07]**

1.8 $\left(y - 2x^3\right)dx - x(1-xy)dy = 0$

[**Ans :** $\dfrac{-y}{x} - x^2 + \dfrac{y^2}{2} = c$] **[MU 1995]**

1.9 $\left(x^2 + y^2 + 1\right)dx - 2xy\,dy = 0$

[**Ans :** $x^2 - y^2 - 1 = cx$] **[MU 2007]**

Rule II: If the given differential equation $M\,dx + N\,dy = 0$ is not exact and if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ is a function of y only, say $f(y)$.

Then the integrating factor is $\text{I.F.} = e^{\int f(y)\,dy}$

The differential equation is reduced to exact differential equation as

$$\left[(\text{I.F.})M\right]dx + \left[(\text{I.F.})N\right]dy = 0$$

Its solution is given as

$$\int_{'y'\text{ constant}} \left[(\text{I.F.})M\right]dx + \int \left[\text{Terms in } \left[(\text{I.F.})N\right], \text{ which are free from } x\right]dy = c$$

EXAMPLES

Example 1.16 Solve $\left(y^4 + 2y\right)dx + \left(xy^3 + 2y^4 - 4x\right)dy = 0.$ **[MU 2003]**

Solution **[6 Marks]**

This equation is of the form

$M\,dx + N\,dy = 0$

Here $M=\left(y^4+2y\right)$ and $N=\left(xy^3+2y^4-4x\right)$

$$\frac{\partial M}{\partial y}=4y^3+2 \quad \text{and} \quad \frac{\partial N}{\partial x}=y^3-4$$

$$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

It is a non-exact differential equation. To find the integrating factor, we have

$$\frac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=\frac{y^3-4-4y^3-2}{y^4+2y}=\frac{-3y^3-6}{y^4+2y}$$

$$=\frac{-3\left(y^3+2\right)}{y\left(y^3+2\right)}=-\frac{3}{y}=f(y) \qquad \text{(say)}$$

$$\therefore \text{I.F.}=e^{\int f(y)dy}=e^{\int-\frac{3}{y}dy}$$

$$=e^{-3\log y}=e^{\log y^{-3}}=\frac{1}{y^3}$$

Now multiplying the given equation by I.F., we get

$$\frac{1}{y^3}\left(y^4+2y\right)dx+\frac{1}{y^3}\left(xy^3+2y^4-4x\right)dy=0, \text{ which is exact}$$

$$\left(y+\frac{2}{y^2}\right)dx+\left(x+2y-\frac{4x}{y^3}\right)dy=0$$

Here $M'=y+\dfrac{2}{y^2}$ and $N'=x+2y-\dfrac{4x}{y^3}$

Its solution is given as

$$\int\limits_{\text{'}y\text{' constant}} M'dx+\int[\text{Terms in } N', \text{ which are free from } x]\,dy=c$$

$$\int\left(y+\frac{2}{y^2}\right)dx+\int 2y\,dy=c$$

$$x\left(y+\frac{2}{y^2}\right)+\frac{2y^2}{2}=c$$

$\therefore$ The solution is $xy+\dfrac{2x}{y^2}+y^2=c$

Example 1.17 Solve $\left(xy^3+y\right)dx+2\left(x^2y^2+x+y^4\right)dy=0.$ **[MU 2009, 12]**

Solution **[6 Marks]**

This equation is of the form

$$M\,dx+N\,dy=0$$

Here $M=xy^3+y$ and $N=2\left(x^2y^2+x+y^4\right)$

$$\frac{\partial M}{\partial y}=3xy^2+1 \quad \text{and} \quad \frac{\partial N}{\partial x}=4xy^2+2$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

It is a non-exact differential equation. To find the integrating factor, we have

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{4xy^2 + 2 - 3xy^2 - 1}{xy^3 + y}$$

$$= \frac{xy^2 + 1}{y\left(1 + xy^2\right)} = \frac{1}{y} = f(y) \qquad \text{(say)}$$

$$\therefore \text{I.F.} = e^{\int f(y)dy} = e^{\int \frac{1}{y}dy}$$
$$= e^{\log y} = y$$

Now multiplying the given equation by I.F., we get

$$y\left(xy^3 + y\right)dx + 2y\left(x^2y^2 + x + y^4\right)dy = 0, \text{which is exact}$$

Here $M' = y\left(xy^3 + y\right)$ and $N' = 2y\left(x^2y^2 + x + y^4\right)$

Its solution is given as

$$\int\limits_{\text{'}y\text{' constant}} M'dx + \int \left[\text{Terms in } N', \text{which are free from } x\right] dy = c$$

$$\int y\left(xy^3 + y\right)dx + \int 2y\left(y^4\right)dy = c$$

$$y^4\frac{x^2}{2} + y^2x + \frac{2y^6}{6} = c$$

$\therefore$ The solution is $\dfrac{x^2y^4}{2} + xy^2 + \dfrac{y^6}{3} = c$

Example 1.18 Solve $\left(3x^2y^4 + 2xy\right)dx + \left(2x^3y^3 - x^2\right)dy = 0.$ **[MU 2010]**

Solution **[6 Marks]**

This equation is of the form

$$M\,dx + N\,dy = 0$$

Here $M = 3x^2y^4 + 2xy$ and $N = 2x^3y^3 - x^2$

$$\frac{\partial M}{\partial y} = 3x^2\left(4y^3\right) + 2x = 12x^2y^3 + 2x \quad \text{and} \quad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

It is a non-exact differential equation. To find the integrating factor, we have

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = -\left[\frac{6x^2y^3 + 4x}{3x^2y^4 + 2xy}\right]$$

$$= -\frac{2}{y}\frac{\left(3x^2y^3 + 2x\right)}{\left(3x^2y^3 + 2x\right)} = -\frac{2}{y} = f(y) \qquad \text{(say)}$$

$\therefore$ I.F. $= e^{\int f(y)dy} = e^{\int -\frac{2}{y}dy}$

$$= e^{-2\log y} = e^{\log y^{-2}} = \frac{1}{y^2}$$

Now multiplying the given equation by I.F., we get

$$\frac{1}{y^2}\left(3x^2y^4 + 2xy\right)dx + \frac{1}{y^2}\left(2x^3y^3 - x^2\right)dy = 0, \text{ which is exact}$$

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0$$

Here $M' = 3x^2y^2 + \dfrac{2x}{y}$ and $N' = 2x^3y - \dfrac{x^2}{y^2}$

Its solution is given as

$$\int\limits_{\text{'}y\text{' constant}} M'dx + \int [\text{Terms in } N', \text{ which are free from } x]\, dy = c$$

$$\int\left(3x^2y^2 + \frac{2x}{y}\right)dx + \int 0\, dy = c$$

$$3y^2\frac{x^3}{3} + \frac{2}{y}\frac{x^2}{2} = c$$

$$x^3y^2 + \frac{x^2}{y} = c$$

$\therefore$ The solution is $x^3y^3 + x^2 = cy$

Example 1.19 Solve $\left(2xy^4e^y + 2xy^3 + y\right)dx + \left(x^2y^4e^y - x^2y^2 - 3x\right)dy = 0.$ **[MU 2005, 10]**

Solution **[6 Marks]**

This equation is of the form

$$M\,dx + N\,dy = 0$$

Here $M = 2xy^4e^y + 2xy^3 + y$ and $N = x^2y^4e^y - x^2y^2 - 3x$

$$\frac{\partial M}{\partial y} = 2x\left[y^4e^y + e^y 4y^3\right] + 2x\left[3y^2\right] + 1$$

$$= 2xy^4e^y + 8xy^3e^y + 6xy^2 + 1$$

$$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

It is a non-exact differential equation. To find the integrating factor, we have

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \frac{2xy^4e^y - 2xy^2 - 3 - 2xy^4e^y - 8xy^3e^y - 6xy^2 - 1}{2xy^4e^y + 2xy^3 + y}$$

$$= \frac{-8xy^2 - 8xy^3e^y - 4}{2xy^4e^y + 2xy^3 + y}$$

$$= \frac{-4\left(2xy^2 + 2xy^3e^y + 1\right)}{y\left(2xy^3e^y + 2xy^2 + 1\right)} = \frac{-4}{y} = f(y) \qquad \text{(say)}$$

$$\therefore \text{I.F.} = e^{\int f(y)dy} = e^{\int -\frac{4}{y}dy} = e^{-4\log y}$$

$$= e^{\log y^{-4}} = \frac{1}{y^4}$$

Now multiplying the given equation by I.F., we get

$$\frac{1}{y^4}\left(2xy^4e^y + 2xy^3 + y\right)dx + \frac{1}{y^4}\left(x^2y^4e^y - x^2y^2 - 3x\right)dy = 0$$

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right)dy = 0, \text{ which is exact}$$

Here $M' = 2xe^y + \frac{2x}{y} + \frac{1}{y^3}$ and $N' = x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}$

Its solution is given as

$$\underset{\text{'}y\text{' constant}}{\int M'dx} + \int \left[\text{Terms in } N', \text{ which are free from } x\right] dy = c$$

$$\int\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \int 0\, dy = c$$

$$2e^y\frac{x^2}{2} + \frac{2}{y}\frac{x^2}{2} + \frac{1}{y^3}x = c$$

$\therefore$ The solution is $x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$

Example 1.20 Solve $y\left(x^2y + e^x\right)dx - e^x dy = 0.$

Solution

This equation is of the form

$$M\,dx + N\,dy = 0$$

Here $M = x^2y^2 + ye^x$ and $N = -e^x$

$$\frac{\partial M}{\partial y} = 2x^2y + e^x \qquad \text{and} \qquad \frac{\partial N}{\partial x} = -e^x$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

It is a non-exact differential equation. To find the integrating factor, we have

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-e^x - 2x^2y - e^x}{x^2y^2 + ye^x} = \frac{-2\left(e^x + x^2y\right)}{y\left(e^x + x^2y\right)}$$

$$= -\frac{2}{y} = f(y) \qquad \text{(say)}$$

$$\therefore \text{I.F.} = e^{\int f(y)dy} = e^{\int -\frac{2}{y}dy}$$

$$= e^{-2\log y} = e^{\log y^{-2}} = \frac{1}{y^2}$$

Now multiplying the given equation by I.F., we get

$$\frac{1}{y^2} y\left(x^2 y + e^x\right)dx - \frac{1}{y^2} e^x dy = 0, \text{ which is exact}$$

Here $M' = \frac{1}{y^2}\left(x^2 y^2 + y e^x\right) = x^2 + \frac{e^x}{y}$ and $N' = -\frac{1}{y^2} e^x$

Its solution is given as

$$\int\limits_{\text{`}y\text{' constant}} M' dx + \int [\text{Terms in } N', \text{ which are free from } x]\, dy = c$$

$$\int \left(x^2 + \frac{e^x}{y}\right)dx + \int 0\, dy = c$$

$$\frac{x^3}{3} + \frac{e^x}{y} = c$$

$\therefore$ The solution is $\frac{x^3}{3} + \frac{e^x}{y} = c$

Example 1.21 Solve $\left(\frac{y}{x}\sec y - \tan y\right)dx + \left(\sec y \log x - x\right)dy = 0.$ **[MU 2006, 08]**

Solution **[6 Marks]**

This equation is of the form

$$M\, dx + N\, dy = 0$$

Here $M = \frac{y}{x}\sec y - \tan y$ and $N = \sec y \log x - x$

$$\frac{\partial M}{\partial y} = \frac{1}{x}\sec y + \frac{y}{x}\sec y \tan y - \sec^2 y \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\sec}{x}$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

It is a non-exact differential equation. To find the integrating factor, we have

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{\frac{\sec y}{x} - 1 - \frac{\sec y}{x} - \frac{y}{x}\sec y \tan y + \sec^2 y}{\frac{y}{x}\sec y - \tan y}$$

$$= \frac{-\frac{y}{x}\sec y \tan y + \tan^2 y}{\frac{y}{x}\sec y - \tan y}$$

$$= \frac{-\tan y\left(\frac{y}{x}\sec y - \tan y\right)}{\frac{y}{x}\sec y - \tan y} = -\tan y = f(y) \text{ (say)}$$

$$\therefore \text{I.F.} = e^{\int f(y)dy} = e^{\int -\tan y dy}$$

$$= e^{\log \cos y} = \cos y$$

Now multiplying the given equation by I.F., we get

$$\cos y\left(\frac{y}{x}\sec y - \tan y\right)dx + \cos y(\sec y \log x - x)dy = 0$$

$$\left(\frac{y}{x} - \sin y\right)dx + (\log x - x\cos y)dy = 0, \text{ which is exact.}$$

Here $M' = \frac{y}{x} - \sin y$ and $N' = \log x - x\cos y$

Its solution is given as

$$\int M'dx + \int [\text{Terms in } N', \text{which are free from } x]dy = c$$

'y'constant

$$\int \left(\frac{y}{x} - \sin y\right)dx + \int 0dy = c$$

$$y\log x - x\sin y = c$$

$\therefore$ The solution is $y\log x - x\sin y = c$

EXERCISES

Solve the following equations:

1.10 $y(xy + e^x)dx - e^x dy = 0$

[**Ans :** $\frac{x^2}{2} + \frac{e^x}{y} = c$] **[MU 1999]**

1.11 $(2xy^2 - y)dx + xdy = 0$

[**Ans :** $x^2 y - x = cy$] **[MU 1994]**

1.12 $(x + 2y^3)\frac{dy}{dx} = y$

[**Ans :** $x - y^3 = cy$]

1.13 $(2x^2 y + e^x)ydx - (e^x + y^3)dy = 0$

[**Ans :** $4x^3 y - 3y^3 + 6e^x = cy$]

Rule III: If the given differential equation $Mdx + Ndy = 0$ is not exact and is of the form $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$, where f_1 and f_2 are functions of the product of xy,

Then the integrating factor is

$$\text{I.F.} = \frac{1}{Mx - Ny}, \text{ where } Mx - Ny \neq 0$$

Now multiplying the given equation by I.F., we get

$$\left[(\text{I.F.})M\right]dx + \left[(\text{I.F.})N\right]dy = 0$$

Its solution is given as

$$\underset{\text{'}y\text{' constant}}{\int}\left[(\text{I.F.})M\right]dx + \int\left[\text{Terms in }\left[(\text{I.F.})N\right]\text{, which are free from } x\right]dy = c$$

EXAMPLES

Example 1.22 Solve $y(1 + xy)dx + x(1 - xy)dy = 0$. **[MU 2005]**

Solution **[6 Marks]**

This equation is of the form

$$f_1(xy)y\,dx + f_2(xy)x\,dy = 0$$

Here $M = y(1 + xy)$ and $N = x(1 - xy)$

$$\therefore \text{I. F.} = \frac{1}{Mx - Ny} = \frac{1}{xy(1+xy) - yx(1-xy)}$$

$$= \frac{1}{xy(1+xy-1+xy)}$$

$$= \frac{1}{2x^2y^2}$$

Now multiplying the given equation by I.F., we get

$$\frac{1}{2x^2y^2}y(1+xy)dx + \frac{1}{2x^2y^2}x(1-xy)dy = 0, \text{ which is exact}$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0$$

Here $M' = \frac{1}{2x^2y} + \frac{1}{2x}$ and $N' = \frac{1}{2xy^2} - \frac{1}{2y}$

Its solution is given as

$$\underset{\text{'}y\text{' constant}}{\int} M'dx + \int\left[\text{Terms in } N'\text{, which are free from } x\right]dy = c$$

$$\int\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\int\left(-\frac{1}{2y}\right)dy=c$$

$$\frac{1}{2y}\left(\frac{x^{-1}}{-1}\right)+\frac{1}{2}\log x-\frac{1}{2}\log y=c$$

$$-\frac{1}{2xy}+\frac{1}{2}\log x-\frac{1}{2}\log y=c$$

$\therefore$ The solution is $-\dfrac{1}{xy}+\log\left(\dfrac{x}{y}\right)=c'$

Example 1.23 Solve $y(1+xy)dx+x\left(1+xy+x^2y^2\right)dy=0$. **[MU 1995]**

Solution **[6 Marks]**

This equation is of the form

$$f_1(xy)\,y\,dx+f_2(xy)\,x\,dy=0$$

Here $M=y(1+xy)$ and $N=x\left(1+xy+x^2y^2\right)$

$$\therefore \text{I.F.}=\frac{1}{Mx-Ny}=\frac{1}{xy(1+xy)-xy\left(1+xy+x^2y^2\right)}$$

$$=\frac{1}{xy\left(1+xy-1-xy-x^2y^2\right)}$$

$$=\frac{1}{-x^3y^3}$$

Now multiplying the given equation by I.F., we get

$$-\frac{1}{x^3y^3}y(1+xy)dx-\frac{1}{x^3y^3}x\left(1+xy+x^2y^2\right)dy=0$$

$$\left(-\frac{1}{x^3y^2}-\frac{1}{x^2y}\right)dx+\left(-\frac{1}{x^2y^3}-\frac{1}{xy^2}-\frac{1}{y}\right)dy=0, \text{ which is exact}$$

Here $M'=-\dfrac{1}{x^3y^2}-\dfrac{1}{x^2y}$ and $N'-\dfrac{1}{x^2y^3}-\dfrac{1}{xy^2}-\dfrac{1}{y}$

Its solution is given as

$$\underset{\text{'}y\text{' constant}}{\int M'dx}+\int\left[\text{Terms in } N'\text{, which are free from } x\right]dy=c$$

$$\int\left(-\frac{1}{x^3y^2}-\frac{1}{x^2y}\right)dx+\int\left(-\frac{1}{y}\right)dy=c$$

$$-\frac{1}{y^2}\left(\frac{x^{-2}}{-2}\right)-\frac{1}{y}\left(\frac{x^{-1}}{-1}\right)-\log y=c$$

$$\frac{1}{2x^2y^2}+\frac{1}{xy}-\log y=c$$

$\therefore$ The solution is $\dfrac{1}{2x^2y^2}+\dfrac{1}{xy}-\log y=c$

Example 1.24 Solve $y(\sin xy+xy\cos xy)dx+x(xy\cos xy-\sin xy)dy=0.$

Solution

This equation is of the form

$$f_1(xy)\,y\,dx+f_2(xy)\,x\,dy=0$$

Here $M=y(\sin xy+xy\cos xy)$ and $N=x(xy\cos xy-\sin xy)$

$$\therefore \text{I.F.}=\frac{1}{Mx-Ny}$$

$$=\frac{1}{xy(\sin xy+xy\cos xy)-xy(xy\cos xy-\sin xy)}$$

$$=\frac{1}{xy(\sin xy+xy\cos xy-xy\cos xy+\sin xy)}$$

$$=\frac{1}{2xy\sin xy}$$

Now multiplying the given equation by I.F., we get

$$\frac{1}{2xy\sin xy}y(\sin xy+xy\cos xy)dx+\frac{1}{2xy\sin xy}x(xy\cos xy-\sin xy)dy=0$$

$$\left(\frac{1}{2x}+\frac{y}{2}\cot xy\right)dx+\left(\frac{x}{2}\cot xy-\frac{1}{2y}\right)dy=0, \text{ which is exact}$$

Here $M'=\dfrac{1}{2x}+\dfrac{y}{2}\cot xy$ and $N'=\dfrac{x}{2}\cot xy-\dfrac{1}{2y}$

Its solution is given as

$$\int\limits_{\text{'}y\text{' constant}} M'dx+\int[\text{Terms in } N'\text{, which are free from } x]\,dy=c$$

$$\int\left(\frac{1}{2x}+\frac{y}{2}\cot xy\right)dx+\int-\frac{1}{2y}dy=c$$

$$\frac{1}{2}\log x+\frac{1}{2}\log(\sin xy)-\frac{1}{2}\log y=\log c$$

$$\frac{1}{2}\log\frac{x\sin xy}{y}=\log c$$

$$\frac{x}{y}\sin xy=c'$$

$\therefore$ The solution is $\frac{x}{y}\sin xy=c'$

Example 1.25 Solve $\frac{dy}{dx}=-\frac{x^2y^3+2y}{2x-2x^3y^2}$. **[MU 1998]**

Solution **[6 Marks]**

We have $\frac{dy}{dx}=-\frac{x^2y^3+2y}{2x-2x^3y^2}$

$\therefore \left(x^2y^3+2y\right)dx+\left(2x-2x^3y^2\right)dy=0$

$\Rightarrow y\left(2+x^2y^2\right)dx+x\left(2-2x^2y^2\right)dy=0$

This equation is of the form

$$f_1(xy)\,y\,dx+f_2(xy)\,x\,dy=0$$

Here $M=\left(2y+x^2y^3\right)$ and $N=\left(2x-2x^3y^2\right)$

$$\therefore \text{I.F.}=\frac{1}{Mx-Ny}=\frac{1}{xy\left(2+x^2y^2\right)-xy\left(2-2x^2y^2\right)}$$

$$=\frac{1}{3x^3y^3}$$

Now multiplying the given equation by I.F., we get

$$\frac{1}{3x^3y^3}y\left(2+x^2y^2\right)dx+\frac{1}{3x^3y^3}x\left(2-2x^2y^2\right)dy=0, \text{ which is exact}$$

$$\left(\frac{2}{3x^3y^2}+\frac{1}{3x}\right)dx+\left(\frac{2}{3x^2y^3}-\frac{2}{3y}\right)dy=c$$

$M'=\frac{2}{3x^3y^2}+\frac{1}{3x}$ and $N'=\frac{2}{3x^2y^3}-\frac{2}{3y}$

Its solution is given as

$$\underset{\text{'}y\text{' constant}}{\int M'dx}+\int\left[\text{Terms in } N'\text{, which are free from } x\right]dy=c$$

$$\int\left(\frac{2}{3x^3y^2}+\frac{1}{3x}\right)dx+\int-\frac{2}{3y}dy=c$$

$$\frac{2}{3y^2}\left(\frac{x^{-2}}{-2}\right)+\frac{1}{3}\log x-\frac{2}{3}\log y=c$$

$$\frac{-1}{3x^2y^2}+\frac{1}{3}\log x-\frac{2}{3}\log y=c$$

$\therefore$ The solution is $\frac{1}{3}\log\frac{x}{y^2}-\frac{1}{3x^2y^2}=c$

Example 1.26 Solve $[xy\sin xy+\cos xy]y\,dx+[xy\sin xy-\cos xy]x\,dy=0$. **[MU 2002, 16]**

Solution **[6 Marks]**

This equation is of the form

$$f_1(xy)y\,dx+f_2(xy)x\,dy=0$$

Here $M=[xy\sin xy+\cos xy]y$ and $N=[xy\sin xy-\cos xy]x$

$$\therefore \text{I.F.}=\frac{1}{Mx-Ny}$$

$$=\frac{1}{xy(xy\sin xy+\cos xy)-xy(xy\sin xy-\cos xy)}$$

$$=\frac{1}{xy(xy\sin xy+\cos xy-xy\sin xy+\cos xy)}$$

$$=\frac{1}{2xy\cos xy}$$

Now multiplying the given equation by I.F., we get

$$\frac{1}{2xy\cos xy}y(xy\sin xy+\cos xy)dx+\frac{1}{2xy\cos xy}x(xy\sin xy-\cos xy)dy=0$$

$$\left(\frac{1}{2x}+\frac{y}{2}\tan xy\right)dx+\left(\frac{x}{2}\tan xy-\frac{1}{2y}\right)dy=0,\text{ which is exact}$$

Here $M'=\frac{1}{2x}+\frac{y}{2}\tan xy$ and $N'=\frac{x}{2}\tan xy-\frac{1}{2y}$

Its solution is given as

$$\underset{\text{`}y\text{' constant}}{\int M'dx}+\int[\text{Terms in } N'\text{, which are free from } x]\,dy=c$$

$$\int\left(\frac{1}{2x}+\frac{y}{2}\tan xy\right)dx+\int\left(-\frac{1}{2y}\right)dy=c$$

$$\frac{1}{2}\log x+\frac{1}{2}\log(\sec xy)-\frac{1}{2}\log y=\log c$$

$$\frac{1}{2}\log\frac{x\sec xy}{y}=\log c$$

$$x \sec xy = c'y$$

$\therefore$ The solution is $x \sec xy = c'y$

Example 1.27 Solve $\left(y - xy^2\right)dx - \left(x + x^2 y\right)dy = 0.$ **[MU 2006, 15]**

Solution **[6 Marks]**

We have $\left(y - xy^2\right)dx - \left(x + x^2 y\right)dy = 0$

$$\Rightarrow (1 - xy)\, y\, dx - (1 + xy)\, x\, dy = 0$$

This equation is of the form

$$f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0$$

Here $M = y - xy^2$ and $N = -x - x^2 y$

$$\therefore \text{ I.F.} = \frac{1}{Mx - Ny}$$

$$= \frac{1}{\left(y - xy^2\right)x - \left[-\left(x + x^2 y\right)\right]y} = \frac{1}{2xy}$$

Now multiplying the given equation by I.F., we get

$$\frac{1}{2xy}\left(y - xy^2\right)dx - \frac{1}{2xy}\left(x + x^2 y\right)dy = 0, \text{ which is exact}$$

$$\left(\frac{1}{2x} - \frac{y}{2}\right)dx - \left(\frac{1}{2y} + \frac{x}{2}\right)dy = 0$$

Here $M' = \dfrac{1}{2x} - \dfrac{y}{2}$ and $N' = -\dfrac{1}{2y} - \dfrac{x}{2}$

Its solution is given as

$$\underset{\text{'}y\text{' constant}}{\int M' dx} + \int \left[\text{Terms in } N', \text{ which are free from } x\right] dy = c$$

$$\int \left(\frac{1}{2x} - \frac{y}{2}\right)dx + \int -\frac{1}{2y}\, dy = c$$

$$\frac{1}{2}\log x - \frac{xy}{2} - \frac{1}{2}\log y = c$$

$$\log\left(\frac{x}{y}\right) - xy = c'$$

$\therefore$ The solution is $\log\left(\dfrac{x}{y}\right) - xy = c'$

Example 1.28 Solve $\left(x^3y^4+x^2y^3+xy^2+y\right)dx+\left(x^4y^3-x^3y^2-x^2y+x\right)dy=0.$ **[MU 2013]**

Solution **[6 Marks]**

We have $\left(x^3y^4+x^2y^3+xy^2+y\right)dx+\left(x^4y^3-x^3y^2-x^2y+x\right)dy=0$

$$\Rightarrow\left(x^3y^3+x^2y^2+xy+1\right)y\,dx+\left(x^3y^3-x^2y^2-xy+1\right)x\,dy=0$$

This equation is of the form

$$f_1(xy)\,y\,dx+f_2(xy)\,x\,dy=0$$

Here $M=x^3y^4+x^2y^3+xy^2+y$ and $N=x^4y^3-x^3y^2-x^2y+x$

$$\therefore\ \text{I.F.}=\frac{1}{Mx-Ny}$$

$$=\frac{1}{x^4y^4+x^3y^3+x^2y^2+xy-x^4y^4+x^3y^3+x^2y^2-xy}$$

$$=\frac{1}{2x^2y^2(xy+1)}$$

Now multiplying the given equation by I.F., we get

$$\frac{\left(x^3y^4+x^2y^3+xy^2+y\right)}{2x^2y^2(xy+1)}dx+\frac{\left(x^4y^3-x^3y^2-x^2y+x\right)}{2x^2y^2(xy+1)}dy=0$$

$$\frac{\left(x^2y^3(xy+1)+y(xy+1)\right)}{2x^2y^2(xy+1)}dx+\frac{\left(x^3y^2(xy-1)-x(xy-1)\right)}{2x^2y^2(xy+1)}dy=0$$

$$\left(\frac{1}{2}y+\frac{1}{2x^2y}\right)dx+\frac{\left(x^3y^2(xy-1)-x(xy-1)\right)}{2x^2y^2(xy+1)}dy=0,\ \text{which is exact}$$

Here $M'=\dfrac{1}{2}y+\dfrac{1}{2x^2y}$ and $N'=\dfrac{\left(x^3y^2(xy-1)-x(xy-1)\right)}{2x^2y^2(xy+1)}$

Its solution is given as

$$\underset{\text{`}y\text{' constant}}{\int M'dx}+\int\left[\text{Terms in } N', \text{ which are free from } x\right]dy=c$$

$$\int\left(\frac{1}{2}y+\frac{1}{2x^2y}\right)dx+\int 0\,dy=c$$

$$\frac{xy}{2}-\frac{1}{2xy}=c$$

$$xy-\frac{1}{xy}=c'$$

$\therefore$ The solution is $xy-\dfrac{1}{xy}=c'$

EXERCISES

Solve the following equations:

1.14 $y\left(xy+2x^2y^2\right)dx+x\left(xy-x^2y^2\right)dy=0$

[**Ans :** $2\log x-\dfrac{1}{xy}-\log y=c$] **[MU 2012]**

1.15 $y\left(1+xy+x^2y^2\right)dx+x\left(1-xy+x^2y^2\right)dy=0$

[**Ans :** $xy+\log\left(\dfrac{x}{y}\right)-\dfrac{1}{xy}=c$]

1.16 $y(2xy+1)dx+x\left(1+2xy-x^3y^3\right)dy=0$

[**Ans :** $\dfrac{1}{x^2y^2}+\dfrac{1}{3x^3y^3}+\log y=c$]

1.17 $y\left(xy+2x^2y^2\right)dx+x\left(xy-x^2y^2\right)dy=0$

[**Ans :** $-\dfrac{1}{3xy}+\dfrac{2}{3}\log x-\dfrac{1}{3}\log y=c$] **[MU 2012]**

Rule IV: If the equation $Mdx + Ndy = 0$ is homogeneous in x and y
Then the integrating factor is

$$\text{I.F.}=\frac{1}{Mx+Ny},\ \text{where}\ \ Mx+Ny\neq 0$$

Now multiplying the given equation by I.F., we get

$$[(I.F.)M]dx+[(I.F.)N]dy=0$$

Its solution is given as

$$\underset{\text{'}y\text{' constant}}{\int}\left[(\text{I.F.})M\right]dx+\int\left[\text{Terms in}\left[(\text{I.F.})N\right],\ \text{which are free from } x\right]dy=c$$

EXAMPLES

Example 1.29 Solve $\left(x^4+y^4\right)dx-xy^3dy=0.$ **[MU 2002, 04]**

Solution **[6 Marks]**

This equation is homogeneous in x and y.

Here $M=x^4+y^4$ and $N=-xy^3$

$$Mx+Ny=x(x^4+y^4)-xy^4=x^5\neq 0$$

$$\therefore \text{I.F.}=\frac{1}{Mx+Ny}=\frac{1}{x^5}$$

Now multiplying the given equation by I.F., we get

$$\frac{1}{x^5}\left(x^4+y^4\right)dx-\frac{1}{x^5}\left(xy^3\right)dy=0$$

$$\left(\frac{1}{x}+\frac{y^4}{x^5}\right)dx-\left(\frac{y^3}{x^4}\right)dy=0,\ \text{which is exact}$$

Here $M'=\dfrac{1}{x}+\dfrac{y^4}{x^5}$ and $N'=-\dfrac{y^3}{x^4}$

Its solution is given as

$$\underset{\text{'y' constant}}{\int M'dx} + \int [\text{Terms in } N', \text{ which are free from } x]\, dy = c$$

$$\int \left(\frac{1}{x} + \frac{y^4}{x^5}\right) dx + \int 0\, dy = c$$

$$\log x + y^4 \frac{x^{-4}}{(-4)} = c$$

$$\log x - \frac{1}{4x^4} y^4 = c$$

$$4x^4 \log x - y^4 = cx^4$$

$\therefore$ The solution is $4x^4 \log x - y^4 = cx^4$

Example 1.30 Solve $\left(3xy^2 - y^3\right)dx - \left(2x^2y - xy^2\right)dy = 0.$

Solution

This equation is homogeneous in x and y.

Here $M = 3xy^2 - y^3$ and $N = -2x^2y + xy^2$

$$\begin{aligned} Mx + Ny &= x[3xy^2 - y^3] + y[-2x^2y + xy^2] \\ &= 3x^2y^2 - xy^3 - 2x^2y^2 + xy^3 \\ &= x^2y^2 \neq 0 \end{aligned}$$

$$\therefore \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{x^2y^2}$$

Now multiplying the given equation by I.F., we get

$$\frac{1}{x^2y^2}\left[3xy^2 - y^3\right]dx - \frac{1}{x^2y^2}\left[2x^2y - xy^2\right]dy = 0$$

$$\left[\frac{3}{x} - \frac{y}{x^2}\right]dx - \left[\frac{2}{y} - \frac{1}{x}\right]dy = 0, \text{ which is exact}$$

Here $M' = \frac{3}{x} - \frac{y}{x^2}$ and $N' = -\frac{2}{y} + \frac{1}{x}$

Its solution is given as

$$\underset{\text{'y' constant}}{\int M'dx} + \int [\text{Terms in } N', \text{ which are free from } x]\, dy = c$$

$$\int \left(\frac{3}{x} - \frac{y}{x^2}\right) dx + \int -\frac{2}{y}\, dy = c$$

$$3\log x - y\left(\frac{x^{-1}}{-1}\right) - 2\log y = c$$

$$3\log x + \frac{y}{x} - 2\log y = c$$

$$\log x^3 - \log y^2 + \frac{y}{x} = c$$

$$\log\frac{x^3}{y^2} = c - \frac{y}{x}$$

$$\frac{x^3}{y^2} = e^{c-y/x}$$

$\therefore$ The solution is $\dfrac{x^3}{y^2} = ce^{-\frac{y}{x}}$

Example 1.31 Solve $y(x+y)dx - x(y-x)dy = 0.$ **[MU 2002]**

Solution **[6 Marks]**

This equation is homogeneous in x and y.

Here $M = y(x+y)$ and $N = -x(y-x)$

$$\begin{aligned} Mx + Ny &= xy(x+y) - xy(y-x) \\ &= x^2y + xy^2 - xy^2 + x^2y \\ &= 2x^2y \neq 0 \end{aligned}$$

$$\therefore \text{I.F.} = \frac{1}{Mx+Ny} = \frac{1}{2x^2y}$$

Now multiplying the given equation by I.F., we get

$$\frac{1}{2x^2y}y(x+y)dx - \frac{1}{2x^2y}x(y-x)dy = 0$$

$$\left(\frac{1}{2x} + \frac{y}{2x^2}\right)dx - \left(\frac{1}{2x} - \frac{1}{2y}\right)dy = 0, \text{ which is exact}$$

Here $M' = \dfrac{1}{2x} + \dfrac{y}{2x^2}$ and $N' = -\dfrac{1}{2x} + \dfrac{1}{2y}$

Its solution is given as

$$\int M'dx + \int[\text{Terms in } N', \text{ which are free from } x]\,dy = c$$

$$\int\left(\frac{1}{2x} + \frac{y}{2x^2}\right)dx + \int\left(\frac{1}{2y}\right)dy = c$$

$$\frac{1}{2}\log x + \frac{y}{2}\left(\frac{x^{-1}}{-1}\right) + \frac{1}{2}\log y = c$$

$$\frac{1}{2}\log x - \frac{y}{2x} + \frac{1}{2}\log y = c$$

$$\frac{1}{2}\log xy = c + \frac{y}{2x}$$

$\therefore$ The solution is $\log\sqrt{xy} = \dfrac{y}{2x} + c$

Example 1.32 Solve $x^2y\,dx - \left(x^3 + y^3\right)dy = 0$.

Solution

This equation is homogeneous in x and y.

Here $M = x^2y$ and $N = -\left(x^3 + y^3\right)$

$$Mx + Ny = x(x^2y) + y(-x^3 - y^3)$$
$$= x^3y - x^3y - y^4 = -y^4 \neq 0$$

$$\therefore \text{I.F.} = \frac{1}{Mx+Ny} = -\frac{1}{y^4}$$

Now multiplying the given equation by I.F., we get

$$-\frac{1}{y^4}\left(x^2y\right)dx + \frac{1}{y^4}\left(x^3 + y^3\right)dy = 0$$

$$\left(-\frac{x^2}{y^3}\right)dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0, \text{ which is exact}$$

Here $M' = -\dfrac{x^2}{y^3}$ and $N' = \dfrac{x^3}{y^4} + \dfrac{1}{y}$

Its solution is given as

$$\underset{\text{'}y\text{' constant}}{\int M'dx} + \int [\text{Terms in } N', \text{which are free from } x]\,dy = c$$

$$\int -\frac{x^2}{y^3}dx + \int \frac{1}{y}dy = c$$

$$-\frac{x^3}{3y^3} + \log y = c$$

$\therefore$ The solution is $-\dfrac{x^3}{3y^3} + \log y = c$

Example 1.33 Solve $\left(x^2y - 2xy^2\right)dx - \left(x^3 - 3x^2y\right)dy = 0$. **[MU 2001, 16]**

Solution **[6 Marks]**

This equation is homogeneous in x and y.

Here $M = x^2y - 2xy^2$ and $N = -x^3 + 3x^2y$

$$Mx + Ny = x(x^2y - 2xy^2) - y(x^3 - 3x^2y)$$
$$= x^3y - 2x^2y^2 - yx^3 + 3x^2y^2$$
$$= x^2y^2 \neq 0$$

$$\therefore \text{I.F.} = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$$

Now multiplying the given equation by I.F., we get

$$\frac{1}{x^2y^2}\left[x^2y - 2xy^2\right]dx + \frac{1}{x^2y^2}\left[-x^3 + 3x^2y\right]dy = 0$$

$$\left[\frac{1}{y}-\frac{2}{x}\right]dx+\left[-\frac{x}{y^2}+\frac{3}{y}\right]dy=0, \text{ which is exact}$$

Here $M'=\frac{1}{y}-\frac{2}{x}$ and $N'=-\frac{x}{y^2}+\frac{3}{y}$

Its solution is given as

$$\underset{\text{'}y\text{' constant}}{\int M'dx}+\int[\text{Terms in } N'\text{, which are free from } x]\,dy=c$$

$$\int\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\frac{3}{y}dy=c$$

$$\frac{x}{y}-2\log x+3\log y=c$$

$$\frac{x}{y}+\log\left(\frac{y^3}{x^2}\right)=c$$

$\therefore$ The solution is $\frac{x}{y}+\log\left(\frac{y^3}{x^2}\right)=c$

EXERCISES

Solve the following equations:

1.18 $(x^2+y^2)dx-(x^2+xy)dy=0$

[**Ans :** $\frac{y}{x}=\log\frac{x}{(x-y)^2}+c$]

1.19 $(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$

[**Ans :** $\frac{x^3}{3}-x^2y-2y^2x+\frac{y^3}{3}=c$] **[MU 2014]**

1.20 $x(x-y)dy+y^2dx=0$

[**Ans :** $cy=e^{y/x}$]

1.21 $(x^3+y^3)dx-xy^2dy=0$

[**Ans :** $cx=c^{y^3/3x^3}$]

1.4 LINEAR DIFFERENTIAL EQUATIONS

A differential equation is said to be linear if the dependent variable y and its derivatives occur only in the first degree and both are not in multiple.

In other words, a differential equation is said to be linear if

1. Every term of its dependent variable and its derivatives occur with no degree higher than first.
2. In no term any two derivatives or a dependent variable are multiplied together.
3. The dependent variable and its derivatives do not appear either in radical sign or in the denominator.

For example, the terms like $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ and $\frac{1}{1+2\frac{dy}{dx}}$ should not be present in the differential equation.

Type I

A differential equation of the form $\frac{dy}{dx}+Py=Q$ is called a linear differential equation of the first order, where P and Q are functions of x alone or constants. Here y is a dependent variable and x is an independent variable.

Working Rule to Find the Solution of $\frac{dy}{dx}+Py=Q$

Step 1: Convert the given equation to standard form of linear differential equation i.e.

$$\frac{dy}{dx}+Py=Q$$

Step 2: Find the integrating factor as

$$\text{Integrating Factor}(\text{I.F.})=e^{\int P dx}$$

Step 3: Then the general solution is given by

$$y(\text{I.F.})=\int Q(\text{I.F.})\,dx+c$$

Note: The coefficient of $\frac{dy}{dx}$ in linear differential equation must be equal to one.

EXAMPLES

Example 1.34 Solve $\left(x^2-1\right)\sin x\frac{dy}{dx}+\left[2x\sin x+\left(x^2-1\right)\cos x\right]y=\left(x^2-1\right)\cos x.$ **[MU 2010]**

Solution **[6 Marks]**

We have $\left(x^2-1\right)\sin x\frac{dy}{dx}+\left[2x\sin x+\left(x^2-1\right)\cos x\right]y=\left(x^2-1\right)\cos x$

Dividing throughout by $\left(x^2-1\right)\sin x,$ we get

$$\frac{dy}{dx}+\left(\frac{2x}{x^2-1}+\frac{\cos x}{\sin x}\right)y=\frac{\cos x}{\sin x}$$

$$\therefore \frac{dy}{dx}+\left(\frac{2x}{x^2-1}+\cot x\right)y=\cot x$$

This is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q$$

Here $P=\frac{2x}{x^2-1}+\cot x$ and $Q=\cot x$

$\therefore$ Integrating factor $(\text{I.F.})=e^{\int P dx}$

$$=e^{\int\left(2x/\left(x^2-1\right)+\cot x\right)dx}$$

$$=e^{\int\left(2x/\left(x^2-1\right)+\cos x\right)dx}$$

$$=e^{\log\left(x^2-1\right)+\log\sin x}$$

$$= e^{\log\{(x^2-1)\sin x\}}$$
$$= (x^2-1)\sin x$$

Its general solution is given by

$$y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + c$$

$$\therefore y(x^2-1)\sin x = \int \cot x(x^2-1)\sin x\,dx + c$$

$$y(x^2-1)\sin x = \int \cos x(x^2-1)\,dx + c$$

$$= (x^2-1)\sin x - 2x(-\cos x) + 2(-\sin x) + c$$

$\therefore$The general solution is

$$y(x^2-1)\sin x = (x^2-1)\sin x + 2x\cos x - 2\sin x + c$$

Example 1.35 Solve $\dfrac{dy}{dx}\cosh x = 2\cosh^2 x\sinh x - y\sinh x$. **[MU 2001, 02, 08]**

Solution **[6 Marks]**

We have $\dfrac{dy}{dx}\cosh x = 2\cosh^2 x\sinh x - y\sinh x$

Dividing throughout by cosh x, we get

$$\frac{dy}{dx} + \frac{\sinh x}{\cosh x}y = 2\sinh x\cosh x$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here $P = \dfrac{\sinh x}{\cosh x}$ and $Q = 2\sinh x\cosh x$

$$\therefore \text{Integrating Factor}(\text{I.F.}) = e^{\int P dx} = e^{\int \sinh x/\cosh x dx}$$
$$= e^{\log\cosh x} = \cosh x$$

Its general solution is given by

$$y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + c$$

$$\therefore y\cosh x = \int 2\sinh x\cosh^2 x\,dx + c$$

Putting cosh $x = t$ $\therefore$ sinh $x\,dx = dt$

$$\therefore y\cosh x = \int 2t^2 dt = \frac{2}{3}t^3 + c$$

$$y\cosh x = \frac{2}{3}\cosh^3 x + c$$

$\therefore$ The general solution is $y\cosh x = \dfrac{2}{3}\cosh^3 x + c$

Example 1.36 Solve $\frac{dy}{dx} + 2y\tan x = \sin x$ at $y = 0, x = \frac{\pi}{3}$.

Solution

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here $P = 2 \tan x$ and $Q = \sin x$

$$\therefore \text{Integrating Factor}(\text{I.F.}) = e^{\int P dx}$$
$$= e^{2\int \tan x\, dx} = e^{2\log \sec x}$$
$$= e^{\log \sec^2 x} = \sec^2 x$$

Its general solution is given by

$$y(\text{I.F.}) = \int Q(\text{I.F.})dx + c$$

$$\therefore y\sec^2 x = \int \sin x(\sec^2 x)dx + c$$
$$= \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx + c$$
$$= \int \tan x \sec x\, dx + c$$

$$\therefore y\sec^2 x = \sec x + c \qquad \left[\because \int \sec x \tan x\, dx = \sec x\right] \tag{1.7}$$

Putting $x = \frac{\pi}{3}, y = 0$ in Eq. (1.7), we get

$$0 = \sec\frac{\pi}{3} + c$$
$$c = -2$$
$$\therefore y\sec^2 x = \sec x - 2$$
$$y = \cos x - 2\cos^2 x$$

$\therefore$ The general solution is $y = \cos x - 2\cos^2 x$

Example 1.37 Solve $x\log x\frac{dy}{dx} + y = 2\log x$.

Solution

We have $x\log x\frac{dy}{dx} + y = 2\log x$

Dividing throughout by $x \log x$, we get

$$\frac{dy}{dx} + \frac{1}{x\log x}y = \frac{2}{x}$$

This is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q$$

Here $P=\dfrac{1}{x\log x}$ and $Q=\dfrac{2}{x}$

$\therefore$ Integrating Factor$(\text{I.F.})=e^{\int P dx}$

$$=e^{\int \frac{1}{x\log x}dx}=e^{\log\log x}=\log x$$

Its general solution is given by

$$y(\text{I.F.})=\int Q(\text{I.F.})\,dx+c$$

$$\therefore\ y\log x=\int \frac{2}{x}\log x\,dx+c$$

$$y\log x=(\log x)^2+c$$

$\therefore$ The general solution is $y\log x=(\log x)^2+c$

Example 1.38 Solve $x(x-1)\dfrac{dy}{dx}-(x-2)y=x^3(2x-1)$. **[MU 2000, 04]**

Solution **[6 Marks]**

We have $x(x-1)\dfrac{dy}{dx}-(x-2)y=x^3(2x-1)$

Dividing throughout by $x(x-1)$, we get

$$\frac{dy}{dx}-\frac{x-2}{x(x-1)}y=\frac{x^2(2x-1)}{x-1}$$

This is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q$$

Here $P=-\dfrac{x-2}{x(x-1)}=-\dfrac{2}{x}+\dfrac{1}{x-1}$ (By partial fraction) and $Q=\dfrac{x^2(2x-1)}{x-1}$

$\therefore$ Integrating Factor$(\text{I. F.})=e^{\int P dx}=e^{\int\left(-\frac{2}{x}+\frac{1}{x-1}\right)dx}$

$$=e^{-2\log x+\log(x-1)}$$

$$=e^{\log\left(\frac{x-1}{x^2}\right)}$$

$$=\frac{x-1}{x^2}$$

Its general solution is given by

$$y(\text{I.F.})=\int Q(\text{I.F.})\,dx+c$$

$$\therefore y\left(\frac{x-1}{x^2}\right)=\int\frac{x^2(2x-1)}{x-1}\cdot\frac{x-1}{x^2}dx+c$$

$$y\left(\frac{x-1}{x^2}\right)=\int(2x-1)dx+c=x^2-x+c$$

$$y=x^3+\frac{cx^2}{x-1}$$

$\therefore$ The general solution is $y=x^3+\dfrac{cx^2}{x-1}$

Example 1.39 Solve $\dfrac{dy}{dx}+\dfrac{4x}{x^2+1}y=\dfrac{1}{(x^2+1)^3}$. **[MU 2002, 15]**

Solution **[6 Marks]**

We have $\dfrac{dy}{dx}+\dfrac{4x}{x^2+1}y=\dfrac{1}{(x^2+1)^3}$

This is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q$$

Here $P=\dfrac{4x}{x^2+1}$ and $Q=\dfrac{1}{(x^2+1)^3}$

$$\therefore \text{Integrating Factor}(\text{I.F.})=e^{\int P dx}=e^{\int\frac{4x}{x^2+1}dx}=e^{2\log(x^2+1)}$$

$$=(x^2+1)^2$$

Its general solution is given by

$$y(\text{I.F.})=\int Q(\text{I.F.})dx+c$$

$$\therefore y(x^2+1)^2=\int\frac{1}{(x^2+1)^3}(x^2+1)^2dx+c$$

$$y(x^2+1)^2=\int\frac{1}{(x^2+1)}dx+c$$

$$y(x^2+1)^2=\tan^{-1}x+c$$

$\therefore$ The general solution is $y(x^2+1)^2=\tan^{-1}x+c$

Example 1.40 Solve $dr+(2r\cot\theta+\sin 2\theta)d\theta=0$.

Solution

We have $\dfrac{dr}{d\theta}+2\cot\theta\, r=-\sin 2\theta$

This is a linear differential equation of the form

$$\frac{dy}{dx}+P\,y=Q$$

Here $P=2\cot\theta$ and $Q=-\sin 2\theta$

$\therefore$ Integrating Factor $(\text{I.F.})=e^{\int P d\theta}$

$$=e^{\int 2\cot\theta\, d\theta}$$

$$=e^{2\log\sin\theta}=\sin^2\theta$$

Its general solution is given by

$$r(\text{I.F.})=\int Q(\text{I.F.})d\theta+c$$

$$\therefore r\sin^2\theta=-\int \sin 2\theta \sin^2\theta d\theta+c$$

$$=-\int 2\sin^3\theta\cos\theta\, d\theta+c$$

Putting $\sin\theta=t \quad \therefore \cos\theta\, d\theta=dt$

$$=-\int 2t^3 dt+c$$

$$=-\frac{2}{4}t^4+c$$

$$\therefore r\sin^2\theta=-\frac{1}{2}\sin^4\theta+c$$

$\therefore$ The general solution is $r\sin^2\theta=-\frac{1}{2}\sin^4\theta+c$

Note: If the given differential equation is in the form $\frac{dr}{d\theta}+Pr=Q$ then the equation is said to be linear in r.

Type II

A differential equation of the form $\frac{dx}{dy}+P'x=Q'$ is called a linear differential equation of the first order, where P' and Q' are functions of y alone or constants. Here x is a dependent variable and y is an independent variable.

Working Rule to Find the Solution of $\frac{dx}{dy}+P'x=Q'$

Step 1: Convert the given equation to standard form of linear differential equation i.e.

$$\frac{dx}{dy}+P'x=Q'$$

Step 2: Find the integrating factor as

Integrating Factor $(\text{I.F.})=e^{\int P'dy}$

Step 3: Then the general solution is given by

$$x(\text{I.F.}) = \int Q'(\text{I.F.})\,dy + c$$

EXAMPLES

Example 1.41 Solve $(1+y^2)dx = (\tan^{-1}y - x)dy.$ **[MU 2002, 16]**

Solution: **[6 Marks]**

We have $(1+y^2)dx = (\tan^{-1}y - x)dy$

Dividing throughout by $1+y^2$, we get

$$dx = \frac{\tan^{-1}y - x}{1+y^2}dy$$

$$\therefore \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2}$$

This is a linear differential equation of the form

$$\frac{dx}{dy} + P'x = Q'$$

Here $P' = \dfrac{1}{1+y^2}$ and $Q' = \dfrac{\tan^{-1}y}{1+y^2}$

$\therefore$ Integrating Factor $(\text{I.F.}) = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

Its general solution is given by

$$x(\text{I.F.}) = \int Q'(\text{I.F.})\,dy + c$$

$$\therefore\ x e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}dy + c$$

Putting $\tan^{-1}y = t$ $\quad \therefore \dfrac{1}{1+y^2}dy = dt$

$$x e^{\tan^{-1}y} = \int t e^t dt + c$$

$$= t e^t - e^t + c \qquad \text{[Integration by parts]}$$

$$x e^{\tan^{-1}y} = e^t(t-1) + c$$

$$= e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$$

$$\therefore x = \tan^{-1}y - 1 + c e^{-\tan^{-1}y}$$

$\therefore$ The general solution is $x = \tan^{-1}y - 1 + c e^{-\tan^{-1}y}$

Example 1.42 Solve $\dfrac{dy}{dx}+\dfrac{y\log y}{x-\log y}=0.$

Solution

We have $\dfrac{dy}{dx}+\dfrac{y\log y}{x-\log y}=0$

On rearranging the equation

$$\frac{dy}{dx}=-\frac{y\log y}{x-\log y}$$

$$\frac{dx}{dy}=\frac{x-\log y}{-y\log y}$$

$$\frac{dx}{dy}+\frac{1}{y\log y}x=\frac{1}{y}$$

This is a linear differential equation of the form

$$\frac{dx}{dy}+P'x=Q'$$

Here $P'=\dfrac{1}{y\log y}$ and $Q'=\dfrac{1}{y}$

$\therefore$ Integrating Factor $(\text{I.F.})=e^{\int\frac{1}{y\log y}dy}$

$$=e^{\log\log y}=\log y$$

Its general solution is given by

$$x(\text{I.F.})=\int Q'(\text{I.F.})\,dy+c$$

$$\therefore x\log y=\int\frac{1}{y}\log y\,dy+c$$

$$x\log y=\frac{(\log y)^2}{2}+c$$

$\therefore$ The general solution is $x\log y=\dfrac{(\log y)^2}{2}+c$

Example 1.43 Solve $dx+x\,dy=e^{-y}\sec^2 y\,dy.$

Solution

We have $dx+x\,dy=e^{-y}\sec^2 y\,dy$

Dividing by dy

$$\frac{dx}{dy}+x=e^{-y}\sec^2 y$$

This is a linear differential equation of the form

$$\frac{dx}{dy}+P'x=Q'$$

Here $P' = 1$ and $Q' = e^{-y}\sec^2 y$

$\therefore$ Integrating Factor $(\text{I. F.}) = e^{\int P' dy}$

$$= e^{\int dy} = e^{y}$$

Its general solution is given by

$$x(\text{I.F.}) = \int Q'(\text{I.F.})\,dy + c$$

$$\therefore x e^{y} = \int e^{-y}\sec^2 y\, e^{y}\, dy + c$$

$$x e^{y} = \int \sec^2 y\, dy + c$$

$$x e^{y} = \tan y + c$$

$\therefore$ The general solution is $xe^{y} = \tan y + c$.

Example 1.44 Solve $(x + 2y^3)dy = y\,dx$. **[MU 2013]**

Solution **[6 Marks]**

We have $(x + 2y^3)dy = y\,dx$

On rearranging the equation

$$y\frac{dx}{dy} = x + 2y^3$$

$$\frac{dx}{dy} = \frac{x}{y} + 2y^2$$

$$\therefore \frac{dx}{dy} - \frac{1}{y}x = 2y^2$$

This is a linear differential equation of the form

$$\frac{dx}{dy} + P'x = Q'$$

Here $P' = -\dfrac{1}{y}$ and $Q' = 2y^2$

$\therefore$ Integrating Factor $(\text{I.F.}) = e^{\int P' dy}$

$$= e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$$

Its general solution is given by

$$x(\text{I.F.}) = \int Q'(\text{I.F.})\,dy + c$$

$$\therefore x\left(\frac{1}{y}\right) = \int 2y^2 \cdot \frac{1}{y}\,dy + c$$

$$\frac{x}{y} = y^2 + c$$

$\therefore$ The general solution is $x = y^3 + cy$

Example 1.45 Solve $(1+\sin y)\frac{dx}{dy} = [2y\cos y - x(\sec y + \tan y)]$. **[MU 2010]**

Solution **[6 Marks]**

We have $(1+\sin y)\frac{dx}{dy} = [2y\cos y - x(\sec y + \tan y)]$

Dividing by $(1+\sin y)$

$$\frac{dx}{dy} = \frac{2y\cos y}{1+\sin y} - \frac{x(\sec y + \tan y)}{1+\sin y}$$

$$\frac{dx}{dy} + \left(\frac{\sec y + \tan y}{1+\sin y}\right)x = \frac{2y\cos y}{1+\sin y}$$

$$\therefore \frac{dx}{dy} + \sec y\; x = \frac{2y\cos y}{1+\sin y}$$

This is a linear differential equation of the form

$$\frac{dx}{dy} + P'x = Q'$$

Here $P' = \sec y$ and $Q' = \frac{2y\cos y}{1+\sin y}$

$\therefore$ Integrating Factor $(\text{I.F.}) = e^{\int P'dy} = e^{\int \sec y\, dy}$

$$= e^{\log(\sec y + \tan y)} = \sec y + \tan y$$

Its general solution is given by

$$x(\text{I.F.}) = \int Q'(\text{I.F.})dy + c$$

$$x(\sec y + \tan y) = \int \frac{2y\cos y}{1+\sin y}(\sec y + \tan y)dy + c$$

$$= \int 2y\, dy + c$$

$$x(\sec y + \tan y) = y^2 + c$$

$\therefore$ The general solution is $x\ (\sec y + \tan y) = y^2 + c$

Example 1.46 Solve $y\log y\, dx + (x - \log y)dy = 0$.

Solution

We have $y\log y\, dx + (x - \log y)dy = 0$

$$\frac{dx}{dy} + \frac{1}{y\ \log y}x - \frac{1}{y} = 0$$

$$\therefore \frac{dx}{dy}+\frac{1}{y\,\log y}x=\frac{1}{y}$$

This is a linear differential equation of the form

$$\frac{dx}{dy}+P'x=Q'$$

Here $P'=\dfrac{1}{y\,\log y}$ and $Q'=\dfrac{1}{y}$

$\therefore$ Integrating Factor $(\text{I. F.})=e^{\int P'\,dy}$

$$=e^{\int\frac{1}{y\,\log y}dy}=e^{\log\log y}=\log y$$

Its general solution is given by

$$x(\text{I.F.})=\int Q'(\text{I.F.})\,dy+c$$

$$x\cdot\log y=\int\frac{1}{y}\log y\,dy+c$$

$$=\int t\,dt+c=\frac{t^2}{2}+c \qquad \left[\text{Putting } \log y=t \;\therefore\; \frac{1}{y}dy=dt\right]$$

$$=\frac{(\log y)^2}{2}+c$$

$\therefore$ The general solution is $x\cdot\log y=\dfrac{(\log y)^2}{2}+c$

Example 1.47 Solve $(y+1)dx+\left(x-(y+2)e^{y}\right)dy=0$ **[MU 1992]**

Solution **[6 Marks]**

We have $(y+1)dx+\left(x-(y+2)e^{y}\right)dy=0$

$$(y+1)\frac{dx}{dy}=-\left(x-(y+2)e^{y}\right)$$

$$\therefore \frac{dx}{dy}+\frac{1}{y+1}x=\frac{(y+2)e^{y}}{y+1}$$

This is a linear differential equation of the form

$$\frac{dx}{dy}+P'x=Q'$$

Here $P'=\dfrac{1}{y+1}$ and $Q'=\dfrac{(y+2)e^{y}}{y+1}$

$\therefore$ Integrating Factor $(\text{I. F.})=e^{\int P'dy}$

$$=e^{\int\frac{1}{y+1}dy}=e^{\log(y+1)}=(y+1)$$

Its general solution is given by

$$x(\text{I.F.}) = \int Q'(\text{I.F.})\,dy + c$$

$$x(y+1) = \int \frac{(y+2)e^y}{y+1}(y+1)\,dy + c$$

$$= \int (y+2)e^y dy + c$$

$$= (y+2)e^y - 1\cdot e^y + c \qquad \text{[Integration by parts]}$$

$$= (y+1)e^y + c$$

$$(y+1)\left(x - e^y\right) = c$$

$\therefore$ The general solution is $(y+1)\left(x-e^y\right) = c$

Example 1.48 Solve $y^4 dx = \left(x^{-3/4} - y^3 x\right)dy$. **[MU 2015]**

Solution **[6 Marks]**

We have $\quad y^4 dx = \left(x^{-3/4} - y^3 x\right)dy$

$$y^4 \frac{dx}{dy} = x^{-3/4} - y^3 x$$

$$\therefore\ y^4 \frac{dx}{dy} + y^3 x = x^{-3/4}$$

Dividing both the sides by y^4, we get

$$\therefore\ \frac{dx}{dy} + \frac{x}{y} = \frac{x^{-3/4}}{y^4}$$

Multiplying both the sides by $x^{3/4}$, we get

$$\therefore\ x^{3/4}\frac{dx}{dy} + \frac{x^{7/4}}{y} = \frac{1}{y^4}$$

Putting $x^{7/4} = t \ \therefore \dfrac{7}{4}x^{3/4}\dfrac{dx}{dy} = \dfrac{dt}{dy}$

The equation becomes

$$\frac{4}{7}\frac{dt}{dy} + \frac{1}{y}t = \frac{1}{y^4}$$

$$\Rightarrow \frac{dt}{dy} + \frac{7}{4y}t = \frac{7}{4y^4}$$

This is a linear differential equation of the form

$$\frac{dt}{dy} + P't = Q'$$

Here $P' = \dfrac{7}{4y}$ and $Q' = \dfrac{7}{4y^4}$

$\therefore$ Integrating Factor $(\text{I.F.}) = e^{\int P' dy} = e^{\int \frac{7}{4y} dy}$

$$= e^{7/4 \log y} = y^{7/4}$$

Its general solution is given by

$$t(\text{I.F.}) = \int Q'(\text{I.F.})\,dy + c$$

$$t \cdot y^{7/4} = \int \frac{7}{4y^4} \cdot y^{7/4} dy + c$$

$$t \cdot y^{7/4} = \frac{7}{4}\int y^{-9/4}\ dy + c$$

$$t \cdot y^{7/4} = \frac{7}{4}\frac{y^{-5/4}}{\left(-\frac{5}{4}\right)} + c$$

$$\therefore x^{7/4} y^{7/4} = -\frac{7}{5} y^{-5/4} + c$$

$\therefore$ The general solution is $x^{7/4} y^{7/4} = -\frac{7}{5} y^{-5/4} + c$

Example 1.49 Solve $(x + y + 1)\dfrac{dy}{dx} = 1.$

Solution

We have $(x + y + 1)\dfrac{dy}{dx} = 1$

On rearranging the equation

$$\frac{dy}{dx} = \frac{1}{x + y + 1}$$

$$\frac{dx}{dy} = x + y + 1$$

$$\frac{dx}{dy} - x = y + 1$$

This is a linear differential equation of the form

$$\frac{dx}{dy} + P'x = Q'$$

Here $P' = -1 \qquad Q' = y + 1$

$\therefore$ Integrating Factor $(\text{I.F.}) = e^{\int P' dy}$

$$= e^{-\int dy} = e^{-y}$$

Its general solution is given by

$$x(\text{I.F.}) = \int Q'(\text{I.F.})\,dy + c$$

$$x e^{-y} = \int (y + 1) e^{-y} dy + c$$

$$= -(y+1)e^{-y} - 1 \cdot e^{-y} + c \qquad \text{[Integration by parts]}$$

$$\therefore \; x = -(y+1) - 1 + c\,e^{y}$$

$$x = -y - 2 + c\,e^{y}$$

$\therefore$ The general solution is $x + y + 2 = ce^{y}$.

EXERCISES

Solve the following equations:

1.22 $\dfrac{dy}{dx} + \left(\dfrac{1-2x}{x^2}\right)y = 1$

[**Ans:** $y = x^2 + ce^{\frac{1}{x}} \cdot x^2$] **[MU 1990]**

1.23 $\sin 2x \dfrac{dy}{dx} = y + \tan x$

[**Ans:** $y = c\sqrt{\tan x} + \tan x$]

1.24 $x\,dy - (y - x)dx = 0$

[**Ans:** $y + x\log x = cx$]

1.25 $\dfrac{dy}{dx} + \dfrac{y}{1-x} = x^2 - x$

[**Ans:** $2y = (1-x)(c^2 - x^2)$]

1.26 $x(1-x^2)\dfrac{dy}{dx} + (2x^2 - 1)y = x^3$

[**Ans:** $y = \tan x + cx\sqrt{1-x^2}$]

1.27 $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

[**Ans:** $y = \tan x - 1 + ce^{-\tan x}$]

1.28 $x(x-1)\dfrac{dy}{dx} - y = x^2(x-1)^2$

[**Ans:** $\dfrac{xy}{1-x} + \dfrac{x^3}{3} = c$]

1.29 $(1 + x + xy^2)dy + (y + y^3)dx = 0$

[**Ans:** $xy + \tan^{-1} y = c$] **[MU 1993, 95]**

1.5 EQUATIONS REDUCIBLE TO LINEAR DIFFERENTIAL EQUATIONS

Type I

The equation of the type $f'(y)\dfrac{dy}{dx} + P\,f(y) = Q$, where P and Q are the functions of x, can be reduced to linear form by substitution.

Putting $f(y) = u$ then $f'(y)\dfrac{dy}{dx} = \dfrac{du}{dx}$

$\therefore$ The equation reduces to $\dfrac{du}{dx} + Pu = Q$

Integrating Factor $(\text{I.F.}) = e^{\int P\,dx}$

The above equation is a linear differential equation in u. Its general solution is given by

$$u(\text{I.F.}) = \int Q(\text{I.F.})\,dx + c$$

EXAMPLES

Example 1.50 Solve $\frac{dy}{dx} = e^{x-y}\left(e^x - e^y\right)$. **[MU 2001, 02, 03, 11]**

Solution **[6 Marks]**

We have $\frac{dy}{dx} = e^{x-y}\left(e^x - e^y\right)$

$$\therefore \frac{dy}{dx} = \frac{e^x}{e^y}\left(e^x - e^y\right)$$

$$e^y \frac{dy}{dx} = e^{2x} - e^x e^y$$

$$e^y \frac{dy}{dx} + e^x e^y = e^{2x}$$

Putting $e^y = u \;\therefore\; e^y \frac{dy}{dx} = \frac{du}{dx}$

$$\Rightarrow \frac{du}{dx} + e^x u = e^{2x}$$

This equation is a linear differential equation in u.

$$\frac{du}{dx} + Pu = Q$$

Here $P = e^x$ and $Q = e^{2x}$

$\therefore$ Integrating Factor $(\text{I.F.}) = e^{\int P dx} = e^{\int e^x dx}$

$$= e^{e^x}$$

Its general solution is given by

$$u(\text{I.F.}) = \int Q(\text{I.F.})dx + c$$

$$\therefore u e^{e^x} = \int e^{2x} e^{e^x} dx + c$$

Putting $e^x = t \quad \therefore\; e^x dx = dt$

$$\Rightarrow u e^{e^x} = \int t e^t dt + c$$

$$= t e^t - e^t + c = e^t(t-1) + c$$

$$\therefore\; e^y e^{e^x} = e^{e^x}\left(e^x - 1\right) + c$$

$\therefore$ The general solution is $e^y = e^x - 1 + c e^{-e^x}$

Example 1.51 Solve $\sec y \frac{dy}{dx} + 2x \sin y = 2x \cos y$.

Solution

We have $\sec y \frac{dy}{dx} + 2x \sin y = 2x \cos y$

Dividing throughout by $\cos y$, we get

$$\sec^2 y\frac{dy}{dx}+2x\tan y=2x$$

Putting $\tan y=u$ $\quad\therefore \sec^2 y\frac{dy}{dx}=\frac{du}{dx}$

$$\Rightarrow \frac{du}{dx}+2xu=2x$$

This equation is a linear differential equation in *u*.

$$\frac{du}{dx}+Pu=Q$$

Here $P=2x$ and $Q=2x$

$\therefore$ Integrating Factor$(\text{I.F.})=e^{\int P dx}$

$$=e^{\int 2x\,dx}=e^{x^2}$$

Its general solution is given by

$$u(\text{I.F.})=\int Q(\text{I.F.})dx+c$$

$$\therefore ue^{x^2}=\int 2xe^{x^2}dx+c$$

Putting $x^2=t$ $\quad\therefore\ 2x\,dx=dt$

$$\Rightarrow ue^{x^2}=\int e^t dt+c$$

$$=e^t+c$$

$$\therefore\ \tan y\,e^{x^2}=e^{x^2}+c$$

$\therefore$ The general solution is $\tan y=1+ce^{-x^2}$

Example 1.52 Solve $\frac{dy}{dx}+\frac{1}{x}\tan y=\frac{1}{x^2}\tan y\sin y.$

Solution

We have $\frac{dy}{dx}+\frac{1}{x}\tan y=\frac{1}{x^2}\tan y\sin y$

Dividing throughout by $\tan y\sin y,$ we get

$$\cot y\,\text{cosec}\,y\frac{dy}{dx}+\text{cosec}\,y\frac{1}{x}=\frac{1}{x^2}$$

Putting $\text{cosec}\,y=u$ $\quad\therefore -\text{cosec}\,y\cot y\frac{dy}{dx}=\frac{du}{dx}$

$$\Rightarrow -\frac{du}{dx}+\frac{1}{x}u=\frac{1}{x^2}$$

$$\frac{du}{dx}-\frac{1}{x}u=-\frac{1}{x^2}$$

This equation is linear differential equation in *u*.

$$\frac{du}{dx}+Pu=Q$$

Here $P=-\frac{1}{x}$ and $Q=-\frac{1}{x^2}$

$\therefore$ Integrating Factor $(\text{I.F.})=e^{\int P dx}=e^{-\int \frac{1}{x}dx}$

$$=e^{-\log x}=\frac{1}{x}$$

Its general solution is given by

$$u(\text{I.F.})=\int Q(\text{I.F.})dx+c$$

$$\therefore u\cdot\frac{1}{x}=-\int\frac{1}{x^2}\cdot\frac{1}{x}dx+c$$

$$\frac{u}{x}=-\int\frac{1}{x^3}dx+c$$

$$\frac{\operatorname{cosec} y}{x}=\frac{1}{2x^2}+c$$

$\therefore\ 2x\operatorname{cosec} y=1+c\,2x^2$

$\therefore$ The general solution is $2x\operatorname{cosec} y=1+c\,2x^2$

Example 1.53 Solve $3y^2\frac{dy}{dx}+2y^3x=4x^3e^{x^2}$.

Solution

We have $3y^2\frac{dy}{dx}+2y^3x=4x^3e^{x^2}$

Putting $y^3=u$ $\therefore 3y^2\frac{dy}{dx}=\frac{du}{dx}$

$$\Rightarrow \frac{du}{dx}+2xu=4x^3e^{x^2}$$

This equation is a linear differential equation in *u*.

$$\frac{du}{dx}+Pu=Q$$

Here $P=2x$ and $Q=4x^3e^{x^2}$

$\therefore$ Integrating Factor $(\text{I.F.})=e^{\int P dx}$

$$=e^{\int 2x\,dx}=e^{x^2}$$

Its general solution is given by

$$u(\text{I.F.})=\int Q(\text{I.F.})dx+c$$

$$\therefore u\left(e^{x^2}\right)=\int 4x^3e^{x^2}e^{x^2}dx+c$$

Putting $x^2=t$ $\therefore 2x\,dx=dt$

$$\Rightarrow ue^{x^2} = \int 2t\,e^{2t}dt + c$$

$$= 2\left[t\left(\frac{e^{2t}}{2}\right) - (1)\left(\frac{e^{2t}}{4}\right)\right] + c \qquad \text{[Integration by parts]}$$

$$= 2\left[\frac{t\,e^{2t}}{2} - \frac{e^{2t}}{4}\right] + c$$

$$= 2\left[\frac{x^2e^{2x^2}}{2} - \frac{e^{2x^2}}{4}\right] + c$$

$$= 2\left[\frac{2x^2e^{2x^2} - e^{2x^2}}{4}\right] + c$$

$$\therefore y^3e^{x^2} = \frac{e^{2x^2}\left(2x^2-1\right)}{2} + c$$

$\therefore$ The general solution is $y^3e^{x^2} = \dfrac{e^{2x^2}\left(2x^2-1\right)}{2} + c$

Example 1.54 Solve $x\dfrac{dy}{dx} - 1 = xe^{-y}$.

Solution

We have $x\dfrac{dy}{dx} - 1 = xe^{-y}$

Dividing throughout by xe^{-y}

$$e^y\frac{dy}{dx} - \frac{1}{x}e^y = 1$$

Putting $e^y = u$ $\qquad \therefore e^y\dfrac{dy}{dx} = \dfrac{du}{dx}$

$$\Rightarrow \frac{du}{dx} - \frac{1}{x}u = 1$$

This equation is a linear differential equation in u.

$$\frac{du}{dx} + Pu = Q$$

Here $P = -\dfrac{1}{x}$ $\qquad$ and $\qquad Q = 1$

$$\therefore \text{Integrating Factor (I.F.)} = e^{\int P dx} = e^{-\int \frac{1}{x}dx}$$

$$= e^{-\log x} = \frac{1}{x}$$

Its general solution is given by

$$u(\text{I.F.}) = \int Q(\text{I.F.})\,dx + c$$

$$\therefore \frac{u}{x} = \int \frac{1}{x} dx + c$$

$$\frac{u}{x} = \log x + c$$

$$\frac{e^y}{x} = \log x + c$$

$$\therefore e^y = x(\log x + c)$$

$\therefore$ The general solution is $e^y = x(\log x + c)$

Example 1.55 Solve $\cos y - x\sin y \dfrac{dy}{dx} = \sec^2 x$.

Solution

We have $\cos y - x\sin y \dfrac{dy}{dx} = \sec^2 x$

$$\therefore -x\sin y \frac{dy}{dx} + \cos y = \sec^2 x$$

Putting $\cos y = u$ $\qquad \therefore -\sin y \dfrac{dy}{dx} = \dfrac{du}{dx}$

$$\Rightarrow x\frac{du}{dx} + u = \sec^2 x$$

Dividing throughout by x

$$\frac{du}{dx} + \frac{1}{x}u = \frac{\sec^2 x}{x}$$

This equation is a linear differential equation in u.

$$\frac{du}{dx} + Pu = Q$$

Here $P = \dfrac{1}{x}$ and $Q = \dfrac{\sec^2 x}{x}$

$$\therefore \text{Integrating Factor (I.F.)} = e^{\int P dx} = e^{\int \frac{1}{x} dx}$$

$$= e^{\log x} = x$$

Its general solution is given by

$$u(\text{I.F.}) = \int Q(\text{I.F.})dx + c$$

$$\therefore ux = \int \frac{\sec^2 x}{x} x\, dx + c$$

$$ux = \tan x + c$$

$\therefore x\cos y = \tan x + c$

$\therefore$ The general solution is $x\cos y = \tan x + c$

Example 1.56 Solve $\left[xy^2 - e^{1/x^3}\right]dx - yx^2dy = 0.$

Solution

We have $\left[xy^2 - e^{1/x^3}\right]dx - yx^2dy = 0$

$$\therefore xy^2 - e^{1/x^3} = yx^2\frac{dy}{dx}$$

$$\frac{xy^2 - e^{1/x^3}}{yx^2} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y}{x} - \frac{e^{1/x^3}}{yx^2}$$

$$y\frac{dy}{dx} - \frac{1}{x}y^2 = -\frac{e^{1/x^3}}{x^2}$$

Putting $y^2 = u$ $\quad\therefore\ 2y\frac{dy}{dx} = \frac{du}{dx}$

$$\frac{1}{2}\frac{du}{dx} - \frac{1}{x}u = -\frac{e^{1/x^3}}{x^2}$$

$$\Rightarrow \frac{du}{dx} - \frac{2}{x}u = -\frac{2e^{1/x^3}}{x^2}$$

This equation is a linear differential equation in u.

$$\frac{du}{dx} + Pu = Q$$

Here $P = -\frac{2}{x}$ and $Q = -\frac{2e^{1/x^3}}{x^2}$

$\therefore$ Integrating Factor $(\text{I.F.}) = e^{\int P dx}$

$$= e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$$

Its general solution is given by

$$u(\text{I.F.}) = \int Q(\text{I.F.})dx + c$$

$$u\frac{1}{x^2} = \int -\frac{2e^{1/x^3}}{x^2}\frac{1}{x^2}dx + c$$

Putting $\frac{1}{x^3} = t$ $\quad\therefore\ -\frac{3}{x^4}dx = dt$

$$u\frac{1}{x^2} = \frac{2}{3}\int e^t dt + c = \frac{2}{3}e^t + c$$

$$\therefore y^2\frac{1}{x^2} = \frac{2}{3}e^{1/x^3} + c$$

$\therefore$ The general solution is $y^2\frac{1}{x^2} = \frac{2}{3}e^{1/x^3} + c$

Example 1.57 Solve $\dfrac{dy}{dx}+x\sin 2y=x^3\cos^2 y$. **[MU 2002, 05, 12, 14]**

Solution **[6 Marks]**

We have $\dfrac{dy}{dx}+x\sin 2y=x^3\cos^2 y$

Dividing both the sides by $\cos^2 y$

$$\sec^2 y\frac{dy}{dx}+2x\tan y=x^3$$

Putting $\tan y=u \qquad \therefore \sec^2 y\dfrac{dy}{dx}=\dfrac{du}{dx}$

$$\Rightarrow \frac{du}{dx}+2xu=x^3$$

This equation is a linear differential equation in *u*.

$$\frac{du}{dx}+Pu=Q$$

Here $P=2x$ and $Q=x^3$

$\therefore$ Integrating Factor $(\text{I.F.})=e^{\int P dx}$

$$=e^{\int 2x dx}=e^{x^2}$$

Its general solution is given by

$$u(\text{I.F.})=\int Q(\text{I.F.})dx+c$$

$$u\left(e^{x^2}\right)=\int x^3e^{x^2}dx+c$$

Putting $x^2=t \qquad \therefore 2xdx=dt$

$$ue^{x^2}=\int te^t\frac{dt}{2}+c=\frac{1}{2}\left[te^t-e^t\right]+c$$

$$\therefore \tan y\, e^{x^2}=\frac{1}{2}e^{x^2}\left[x^2-1\right]+c$$

$\therefore$ The general solution is $\tan y\, e^{x^2}=\dfrac{1}{2}e^{x^2}\left[x^2-1\right]+c$

Type II

The equation of the type $f'(x)\dfrac{dx}{dy}+P'f(x)=Q'$, where P' and Q' are the functions of *y*, can be reduced to linear form by substitution.

Putting $f(x)=u$ then $f'(x)\dfrac{dx}{dy}=\dfrac{du}{dy}$

$\therefore$ The equation reduces to $\dfrac{du}{dy}+P'u=Q'$

Integrating Factor $(\text{I.F.})=e^{\int P' dx}$

The above differential is a linear differential equation in *u*. Its general solution is given by

$$u(\text{I.F.})=\int Q'(\text{I.F.})\,dy+c$$

EXAMPLES

Example 1.58 Solve $\dfrac{dy}{dx}=\dfrac{y^3}{e^{2x}+y^2}$. **[MU 2008]**

Solution **[6 Marks]**

We have
$$\frac{dy}{dx}=\frac{y^3}{e^{2x}+y^2}$$
$$\therefore \frac{dx}{dy}=\frac{e^{2x}+y^2}{y^3}$$
$$\frac{dx}{dy}-\frac{1}{y}=\frac{e^{2x}}{y^3}$$
$$\Rightarrow e^{-2x}\frac{dx}{dy}-e^{-2x}\frac{1}{y}=\frac{1}{y^3}$$

Putting $e^{-2x}=u$ $\quad\therefore -2e^{-2x}\dfrac{dx}{dy}=\dfrac{du}{dy}\Rightarrow e^{-2x}\dfrac{dx}{dy}=-\dfrac{1}{2}\dfrac{du}{dy}$

$$-\frac{1}{2}\frac{du}{dy}-\frac{1}{y}u=\frac{1}{y^3}$$
$$\Rightarrow \frac{du}{dy}+\frac{2}{y}u=\frac{-2}{y^3}$$

This equation is a linear differential equation in u.

$$\frac{du}{dy}+P'u=Q'$$

Here $P'=\dfrac{2}{y}$ and $Q'=\dfrac{-2}{y^3}$

$\therefore$ Integrating Factor $(\text{I.F.})=e^{\int P'dy}=e^{\int\frac{2}{y}dy}$
$$=e^{2\log y}=e^{\log y^2}=y^2$$

Its general solution is given by

$$u(\text{I.F.})=\int Q'(\text{I.F.})\,dy+c$$
$$u\,y^2=\int\frac{-2}{y^3}y^2dy+c$$
$$=-2\log y+c$$

$\therefore e^{-2x}y^2+2\log y=c$

$\therefore$ The general solution is $e^{-2x}y^2+2\log y=c$

Example 1.59 Solve $y\dfrac{dx}{dy}=x+yx^2\log y$.

Solution

We have $y\dfrac{dx}{dy}=x+yx^2\log y$

Dividing both the sides by yx^2

$$\frac{1}{x^2}\frac{dx}{dy} = \frac{1}{x}\cdot\frac{1}{y} + \log y$$

$$\therefore \frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}\cdot\frac{1}{y} = \log y$$

Putting $-\frac{1}{x} = u$ $\qquad \therefore \frac{1}{x^2}\frac{dx}{dy} = \frac{du}{dy}$

$$\Rightarrow \frac{du}{dy} + \frac{1}{y}u = \log y$$

This equation is a linear differential equation in u.

$$\frac{du}{dy} + P'u = Q'$$

Here $P' = \frac{1}{y}$ and $Q' = \log y$

$$\therefore \text{Integrating Factor (I.F.)} = e^{\int P'dy} = e^{\int \frac{1}{y}dy}$$

$$= e^{\log y} = y$$

Its general solution is given by

$$u(\text{I.F.}) = \int Q'(\text{I.F.})dy + c$$

$$\therefore uy = \int y\log y\, dy + c$$

$$= \log y\frac{y^2}{2} - \int \frac{y^2}{2}\cdot\frac{1}{y}dy + c$$

$$= \log y\frac{y^2}{2} - \frac{y^2}{4} + c$$

$$-\frac{y}{x} = \log y\frac{y^2}{2} - \frac{y^2}{4} + c$$

$\therefore$ The general solution is $\frac{y}{x} + \frac{y^2}{2}\log y - \frac{y^2}{4} = c$

Example 1.60 Solve $y\sin x\frac{dx}{dy} - \cos x = 2y^3\cos^2 x$.

Solution

We have $y\sin x\frac{dx}{dy} - \cos x = 2y^3\cos^2 x$

Dividing both the sides by $y\cos^2 x$

$$\tan x\sec x\frac{dx}{dy} - \frac{\sec x}{y} = 2y^2$$

Putting $\sec x = u$ $\qquad \therefore \sec x\tan x\frac{dx}{dy} = \frac{du}{dy}$

$$\Rightarrow \frac{du}{dy} - \frac{u}{y} = 2y^2$$

This equation is a linear differential equation in u.

$$\frac{du}{dy} + P'u = Q'$$

Here $P' = -\frac{1}{y}$ and $Q' = 2y^2$

$$\therefore \text{Integrating Factor (I.F.)} = e^{\int P'dy} = e^{-\int \frac{1}{y}dy}$$

$$= e^{-\log y} = \frac{1}{y}$$

Its general solution is given by

$$u(\text{I.F.}) = \int Q'(\text{I.F.})\,dy + c$$

$$\therefore \frac{u}{y} = \int 2y^2 \cdot \frac{1}{y}\,dy + c$$

$$\frac{u}{y} = y^2 + c$$

$$\frac{\sec x}{y} = y^2 + c$$

$$\sec x = y^3 + cy$$

$\therefore$ The general solution is $\sec x = y^3 + cy$

Example 1.61 Solve $\frac{dx}{dy} = e^{y-x}\left(e^y - e^x\right)$.

Solution

We have $\frac{dx}{dy} = e^{y-x}\left(e^y - e^x\right)$

$$\frac{dx}{dy} = \frac{e^y}{e^x}\left(e^y - e^x\right)$$

$$e^x\frac{dx}{dy} = e^{2y} - e^y e^x$$

$$e^x\frac{dx}{dy} + e^y e^x = e^{2y}$$

Putting $e^x = u$ $\quad\therefore e^x\frac{dx}{dy} = \frac{du}{dy}$

$$\Rightarrow \frac{du}{dy} + e^y u = e^{2y}$$

This equation is a linear differential equation in u.

$$\frac{du}{dy} + P'u = Q'$$

Here $P' = e^y$ and $Q' = e^{2y}$

$\therefore$ Integrating Factor $(\text{I.F.}) = e^{\int P' dy}$

$$= e^{\int e^y dy} = e^{e^y}$$

Its general solution is given by

$$u(\text{I.F.}) = \int Q'(\text{I.F.})\,dy + c$$

$$\therefore ue^{e^y} = \int e^{2y} e^{e^y}\,dy + c$$

Putting $e^y = t$ $\qquad\therefore\ e^y dy = dt$

$$ue^{e^y} = \int t e^t dt + c = te^t - e^t + c$$

$$= e^t(t-1) + c$$

$$e^x e^{e^y} = e^{e^y}(e^y - 1) + c$$

$$\therefore\ e^x = e^y - 1 + ce^{-e^y}$$

$\therefore$ The general solution is $e^x = e^y - 1 + ce^{-e^y}$

Example 1.62 Solve $e^x(x+1)dx + (y^2e^{2y} - xe^x)dy = 0.$

Solution

We have $e^x(x+1)dx + (y^2e^{2y} - xe^x)dy = 0$

$$e^x(x+1)\frac{dx}{dy} - xe^x = -y^2e^{2y}$$

Putting $xe^x = u$

$$\therefore (xe^x + e^x)\frac{dx}{dy} = \frac{du}{dy}$$

$$e^x(x+1)\frac{dx}{dy} = \frac{du}{dy}$$

$$\Rightarrow \frac{du}{dy} - u = -y^2e^{2y}$$

This equation is a linear differential equation in *u*.

$$\frac{du}{dy} + P'u = Q'$$

Here $P' = -1$ and $Q' = -y^2e^{2y}$

$\therefore$ Integrating Factor $(\text{I.F.}) = e^{\int P' dy}$

$$= e^{-\int dy} = e^{-y}$$

Its general solution is given by

$$u(\text{I.F.}) = \int Q'(\text{I.F.})\,dy + c$$

$$ue^{-y} = -\int y^2e^{2y}e^{-y}\,dy + c$$

$$= -\int y^2 e^y\,dy + c$$

$$= -\left[y^2e^y - \int e^y \cdot 2y\,dy\right] + c$$

$$= -y^2e^y + 2\left[e^y y - \int e^y dy\right] + c$$

$$= -y^2e^y + 2ye^y - 2e^y + c$$

$$\therefore xe^xe^{-y} = -y^2e^y + 2ye^y - 2e^y + c$$

$\therefore$ The general solution is $xe^xe^{-y} = -y^2e^y + 2ye^y - 2e^y + c$

EXERCISES

Solve the following equations:

1.30 $x\cos y\dfrac{dy}{dx} - \sin y = x\sin^2 y$

[**Ans:** $\operatorname{cosec} y + x(\log x + c) = 0$]

1.31 $\sec^2 y\dfrac{dy}{dx} + 2\tan x\tan y = \sin x$

[**Ans:** $\sec^2 x\tan y = \sec x + c$]

1.32 $y\dfrac{dy}{dx} + \dfrac{4x}{3} - \dfrac{y^2}{3x} = 0$

[**Ans:** $y^2x^{-2/3} + 2x^{4/3} = c$] **[MU 2002]**

1.33 $\tan y\dfrac{dy}{dx} + \tan x(1 - \cos y) = 0$

[**Ans:** $\sec y = 1 + c\cdot\cos x$]

1.34 $\dfrac{dy}{dx} + \left(2x\tan^{-1}y - x^3\right)\left(1 + y^2\right) = 0$

[**Ans:** $\tan^{-1}y = \dfrac{x^2 - 1}{2} + ce^{-x^2}$] **[MU 2006]**

1.35 $\dfrac{dy}{dx} + x^3\sin^2 y + x\sin 2y = x^3$

[**Ans:** $\tan y\cdot e^{x^2} = \dfrac{1}{2}e^{x^2}\left(x^2 - 1\right) + c$] **[MU 1987]**

Type III: Bernoulli's Equation

A differential equation of the form $\dfrac{dy}{dx} + Py = Qy^n$ is called a Bernoulli's equation. Here P and Q are functions of x alone or constants and n is a real number.

The above equation can be made a linear differential equation by dividing both the sides by y^n.

$$\therefore y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \tag{1.8}$$

Putting $y^{1-n} = u$

$$\therefore (1-n)y^{-n}\frac{dy}{dx} = \frac{du}{dx}$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}$$

$$\Rightarrow \frac{1}{1-n}\frac{du}{dx} + Pu = Q$$

$$\therefore \frac{du}{dx} + (1-n)Pu = (1-n)Q$$

It is a linear differential equation in u. Its general solution is given by

$$u(\text{I.F.}) = \int Q(\text{I.F.})\ dx + c$$

Similarly $\dfrac{dx}{dy} + P'x = Q'x^n$ is also called a Bernoulli's equation. Here P' and Q' are the functions of y alone or constants. The above equation can be made linear by dividing throughout by x^n.

$$x^{-n}\frac{dx}{dy} + P'x^{1-n} = Q'$$

Put $x^{1-n} = u$ and proceed.

EXAMPLES

Example 1.63 Solve $\dfrac{dy}{dx} = x^3y^3 - xy$. **[MU 2007, 11]**

Solution **[6 Marks]**

We have $\dfrac{dy}{dx} = x^3y^3 - xy$

Dividing throughout by y^3

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}x = x^3 \quad [\text{Bernoulii's equation}]$$

Putting $\dfrac{1}{y^2} = u$

$$\therefore -\frac{2}{y^3}\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{du}{dx}$$

$$\Rightarrow -\frac{1}{2}\frac{du}{dx} + xu = x^3$$

$$\frac{du}{dx} - 2xu = -2x^3$$

This equation is a linear differential equation in u.

$$\frac{du}{dx} + Pu = Q$$

Here $P = -2x$ and $Q = -2x^3$

$$\therefore \text{ Integrating Factor}(\text{I.F.}) = e^{\int P\,dx}$$

$$= e^{-2\int x\,dx} = e^{-x^2}$$

Its general solution is given by

$$u(\text{I.F.}) = \int Q(\text{I.F.})\,dx + c$$

$$ue^{-x^2} = -2\int x^3e^{-x^2}\,dx + c$$

Putting $-x^2 = t$ $\quad\therefore\; -2x\,dx = dt$

$$ue^{-x^2} = -\int t\,e^t\,dt + c = -\left[t\,e^t - e^t\right] + c$$

$$ue^{-x^2} = -e^t\left(t-1\right) + c$$

$$\therefore\; \frac{1}{y^2}e^{-x^2} = -e^{-x^2}\left(-x^2-1\right) + c$$

$\therefore$ The general solution is $\dfrac{1}{y^2}e^{-x^2} = e^{-x^2}\left(x^2+1\right) + c$

Example 1.64 Solve $\dfrac{dy}{dx} = 2y\left(1-2xy\right)$.

Solution

We have $\dfrac{dy}{dx} = 2y\left(1-2xy\right)$

$$\frac{dy}{dx} - 2y = -4xy^2 \quad \text{[Bernoulii's equation]}$$

Dividing both the sides by $\dfrac{1}{y^2}$

$$\therefore\; \frac{1}{y^2}\frac{dy}{dx} - \frac{2}{y} = -4x$$

Putting $-\dfrac{1}{y} = u$ $\qquad \therefore\; \dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{du}{dx}$

$$\Rightarrow \frac{du}{dx} + 2u = -4x$$

This equation is a linear differential equation in u.

$$\frac{du}{dx} + Pu = Q$$

Here $P = 2$ and $Q = -4x$

$\therefore$ Integrating Factor $\left(\text{I.F.}\right) = e^{\int P\,dx}$

$$= e^{\int 2\,dx} = e^{2x}$$

Its general solution is given by

$$u\left(\text{I.F.}\right) = \int Q\left(\text{I.F.}\right)\,dx + c$$

$$u\,e^{2x} = -\int 4x\,e^{2x}dx + c$$

$$= -4\left[x\left(\frac{e^{2x}}{2}\right) - 1\cdot\left(\frac{e^{2x}}{4}\right)\right] + c \qquad \text{[Integration by parts]}$$

$$= -4\left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right] + c$$

$$= -2xe^{2x} + e^{2x} + c$$

$$-\frac{1}{y}e^{2x} = -2xe^{2x} + e^{2x} + c$$

$$-\frac{1}{y} = -2x + 1 + c\,e^{-2x}$$

$$\therefore\ \frac{1}{y} = 2x - 1 - c\,e^{-2x}$$

$\therefore$ The general solution is $\frac{1}{y} = 2x - 1 - c\,e^{-2x}$

Example 1.65 Solve $x\frac{dy}{dx} + y = y^3x^{n+1}$. **[MU 1992]**

Solution **[6 Marks]**

We have $x\frac{dy}{dx} + y = y^3x^{n+1}$ [Bernoulii's equation]

Dividing both the sides by xy^3

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}\cdot\frac{1}{x} = x^n$$

Putting $\frac{1}{y^2} = u$

$$\therefore \frac{-2}{y^3}\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{du}{dx}$$

$$\Rightarrow -\frac{1}{2}\frac{du}{dx} + \frac{1}{x}u = x^n$$

$$\frac{du}{dx} - \frac{2}{x}u = -2\,x^n$$

This equation is a linear differential equation in *u*.

$$\frac{du}{dx} + Pu = Q$$

Here $P = -\frac{2}{x}$ and $Q = -2\,x^n$

$$\therefore \text{Integrating Factor (I.F.)} = e^{\int P dx} = e^{-\int \frac{2}{x}dx}$$

$$= e^{-2\log x} = \frac{1}{x^2}$$

Its general solution is given by

$$u(\text{I.F.}) = \int Q(\text{I.F.})\ dx + c$$

$$u \cdot \frac{1}{x^2} = -\int 2x^n \cdot \frac{1}{x^2}\, dx + c$$

$$= -\int 2x^{n-2} dx + c$$

$$= -\frac{2x^{n-1}}{n-1} + c$$

$$\frac{1}{x^2 y^2} = -\frac{2x^{n-1}}{n-1} + c$$

$$\therefore\ \frac{n-1}{y^2} = -2x^{n+1} + cx^2$$

$\therefore$ The general solution is $\dfrac{n-1}{y^2} = -2x^{n+1} + cx^2$

Example 1.66 Solve $\dfrac{dz}{dx} + \dfrac{z}{x}\log z = \dfrac{z}{x^2}(\log z)^3$. **[MU 2005]**

Solution **[6 Marks]**

We have $\dfrac{dz}{dx} + \dfrac{z}{x}\log z = \dfrac{z}{x^2}(\log z)^3$ [Bernoulii's equation]

Dividing both the sides by $z\,(\log z)^3$

$$\frac{1}{z\,(\log z)^3}\frac{dz}{dx} + \frac{1}{(\log z)^2}\frac{1}{x} = \frac{1}{x^2}$$

Putting $\dfrac{1}{(\log z)^2} = u$

$$\therefore\ \frac{-2}{z(\log z)^3}\frac{dz}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{1}{z(\log z)^3}\frac{dz}{dx} = -\frac{1}{2}\frac{du}{dx}$$

$$\Rightarrow -\frac{1}{2}\frac{du}{dx} + \frac{1}{x}u = \frac{1}{x^2}$$

$$\frac{du}{dx} - \frac{2}{x}u = -\frac{2}{x^2}$$

This equation is a linear differential equation in u.

$$\frac{du}{dx} + P\,u = Q$$

Here $P = -\dfrac{2}{x}$ and $Q = -\dfrac{2}{x^2}$

$$\therefore\ \text{Integrating Factor (I.F.)} = e^{\int P\,dx} = e^{-\int \frac{2}{x}dx}$$

$$= e^{-2\log x} = \frac{1}{x^2}$$

Its general solution is given by

$$u(\text{I.F.}) = \int Q(\text{I.F.})\,dx + c$$

$$u \cdot \frac{1}{x^2} = \int -\frac{2}{x^2} \cdot \frac{1}{x^2}\,dx + c$$

$$\frac{1}{(\log z)^2} \cdot \frac{1}{x^2} = -2\left[\frac{x^{-3}}{-3}\right] + c$$

$$\therefore \frac{1}{(\log z)^2} \cdot \frac{1}{x^2} = \frac{2}{3x^3} + c$$

$\therefore$ The general solution is $\dfrac{1}{(\log z)^2} \cdot \dfrac{1}{x^2} = \dfrac{2}{3x^3} + c$

Example 1.67 Solve $y\dfrac{dx}{dy} = x - yx^2 \cos y$. **[MU 1991]**

Solution **[6 Marks]**

We have $y\dfrac{dx}{dy} = x - yx^2 \cos y$

Dividing both the sides by y

$$\frac{dx}{dy} = \frac{x}{y} - x^2 \cos y$$

$$\therefore \frac{dx}{dy} - \frac{1}{y}x = -x^2 \cos y \qquad [\text{Bernoulii's equation}]$$

Now, dividing both the sides by x^2

$$\frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x} \cdot \frac{1}{y} = -\cos y$$

Putting $u = -\dfrac{1}{x}$

$$\therefore \frac{1}{x^2}\frac{dx}{dy} = \frac{du}{dy}$$

$$\Rightarrow \frac{du}{dy} + \frac{1}{y}u = -\cos y$$

This equation is a linear differential equation in u.

$$\frac{du}{dy} + P'u = Q'$$

Here $P' = \dfrac{1}{y}$ and $Q' = -\cos y$

$$\therefore \text{ Integrating Factor } (\text{I.F.}) = e^{\int P'dy} = e^{\int \frac{1}{y}dy}$$

$$= e^{\log y} = y$$

Its general solution is given by

$$u(\text{I.F.}) = \int Q'(\text{I.F.})\,dy + c$$

$$uy = -\int y\cos y\,dy + c$$

$$= -[y\sin y - (1)(-\cos y)] + c$$

$$= -y\sin y - \cos y + c$$

$$-\frac{y}{x} = -y\sin y - \cos y + c$$

$$\therefore \frac{y}{x} = y\sin y + \cos y + c'$$

$\therefore$ The general solution is $\dfrac{y}{x} = y\sin y + \cos y + c'$

Example 1.68 Solve $y\,dx + x\left(1-3x^2y^2\right)dy = 0.$ **[MU 2009]**

Solution **[6 Marks]**

We have $y\,dx + x\left(1-3x^2y^2\right)dy = 0$

The equation can be written as

$$\frac{dx}{dy} + \frac{x}{y} = 3x^3y \qquad [\text{Bernoulii's equation}]$$

Dividing both the sides by x^3

$$\therefore x^{-3}\frac{dx}{dy} + \frac{1}{y}x^{-2} = 3y$$

Putting $x^{-2} = u$

$$\therefore (-2)x^{-3}\frac{dx}{dy} = \frac{du}{dy}$$

$$\Rightarrow -\frac{1}{2}\frac{du}{dy} + \frac{1}{y}u = 3y$$

$$\frac{du}{dy} - \frac{2}{y}u = -6y$$

This equation is a linear differential equation in u.

$$\frac{du}{dy} + P'u = Q'$$

Here $P' = -\dfrac{2}{y}$ and $Q' = -6y$

$$\therefore \text{Integrating Factor}(\text{I.F.}) = e^{\int P'dy} = e^{-\int \frac{2}{y}dy}$$

$$= e^{-2\log y} = \frac{1}{y^2}$$

Its general solution is given by

$$u(\text{I.F.}) = \int Q'(\text{I.F.})\,dy + c$$

$$u \cdot \frac{1}{y^2} = -\int 6y \cdot \frac{1}{y^2}\,dy + c$$

$$\frac{1}{x^2} \cdot \frac{1}{y^2} = -6\log y + c$$

$$\therefore \; \frac{1}{x^2 y^2} + 6\log y = c$$

$\therefore$ The general solution is $\dfrac{1}{x^2 y^2} + 6\log y = c$

Example 1.69 Solve $xy\left(1+xy^2\right)\dfrac{dy}{dx} = 1.$ **[MU 2002, 03, 14]**

Solution **[6 Marks]**

We have $xy\left(1+xy^2\right)\dfrac{dy}{dx} = 1$

$$\frac{dx}{dy} = xy + x^2 y^3$$

$$\Rightarrow \frac{dx}{dy} - xy = x^2 y^3 \qquad [\text{Bernoulii's equation}]$$

Dividing throughout by x^2

$$\therefore \; \frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}y = y^3$$

Putting $-\dfrac{1}{x} = u$

$$\therefore \; \frac{1}{x^2}\frac{dx}{dy} = \frac{du}{dy}$$

$$\Rightarrow \frac{du}{dy} + yu = y^3$$

This equation is a linear differential equation in u.

$$\frac{du}{dy} + P'u = Q'$$

Here $P' = y$ and $Q' = y^3$

$\therefore$ Integrating Factor $(\text{I.F.}) = e^{\int P' dy}$

$$= e^{\int y dy} = e^{y^2/2}$$

Its general solution is given by

$$u(\text{I.F.}) = \int Q'(\text{I.F.})\,dy + c$$

$$\therefore u\,e^{y^2/2} = \int y^3 e^{y^2/2}\,dy + c$$

Putting $\dfrac{y^2}{2} = t$

$\therefore ydy = dt$

$$\therefore \; ue^{y^2/2} = \int 2te^t dt + c$$

$$= 2\left[te^t - e^t\right] + c$$

$$= 2e^t(t-1) + c$$

$$-\frac{1}{x}e^{y^2/2} = 2e^{y^2/2}\left(\frac{y^2}{2} - 1\right) + c$$

$$\therefore -\frac{1}{x} = y^2 - 2 + ce^{-y^2/2}$$

$\therefore$ The general solution is $\dfrac{1}{x} = 2 - y^2 + ce^{-y^2/2}$

Example 1.70 Solve $\left(x^3y^3 - xy\right)dy = dx$.

Solution

We have $\left(x^3y^3 - xy\right)dy = dx$

$$\left(x^3y^3 - xy\right) = \frac{dx}{dy}$$

$$\Rightarrow \frac{dx}{dy} + yx = x^3y^3 \qquad \text{[Bernoulii's equation]}$$

Dividing both the sides by x^3, we get

$$\frac{1}{x^3}\frac{dx}{dy} + \frac{1}{x^2}y = y^3$$

Putting $\dfrac{1}{x^2} = u$

$$\therefore -\frac{2}{x^3}\frac{dx}{dy} = \frac{du}{dy} \qquad \Rightarrow \frac{1}{x^3}\frac{dx}{dy} = -\frac{1}{2}\frac{du}{dy}$$

$$\Rightarrow -\frac{1}{2}\frac{du}{dy} + yu = y^3$$

$$\therefore \frac{du}{dy} - 2yu = -2y^3$$

This equation is a linear differential equation in u.

$$\frac{du}{dy} + P'u = Q'$$

Here $P' = -2y$ and $Q' = -2y^3$

$$\therefore \text{Integrating Factor (I.F.)} = e^{\int P'dy}$$

$$= e^{-2\int ydy} = e^{-y^2}$$

Its general solution is given by

$$u(\text{I.F.}) = \int Q'(\text{I.F.})\,dy + c$$

$$\therefore ue^{-y^2} = -2\int y^3 e^{-y^2}\,dy + c$$

Putting $-y^2 = t \qquad \therefore -2y\,dy = dt$

$$\therefore ue^{-y^2} = -\int te^t\,dt + c$$

$$u\,e^{-y^2} = -[te^t - e^t] + c$$

$$\therefore \frac{1}{x^2}e^{-y^2} = -e^{-y^2}\left(-y^2 - 1\right) + c$$

$\therefore$ The general solution is $\dfrac{1}{x^2}e^{-y^2} = e^{-y^2}\left(y^2 + 1\right) + c$

EXAMPLES ON EQUATIONS REDUCIBLE TO BERNOULI'S FORM

Example 1.71 Solve $\dfrac{dy}{dx} = 1 - x(y - x) - x^3(y - x)^2$. **[MU 2000, 02, 08]**

Solution **[6 Marks]**

We have $\dfrac{dy}{dx} = 1 - x(y - x) - x^3(y - x)^2$

Putting $y - x = u$

$$\therefore \frac{dy}{dx} - 1 = \frac{du}{dx}$$

$$\Rightarrow \frac{du}{dx} + xu = -x^3u^2$$

Dividing both the sides by $-u^2$

$$-\frac{1}{u^2}\frac{du}{dx} - \frac{x}{u} = x^3$$

Putting $\dfrac{1}{u} = t$

$$\therefore -\frac{1}{u^2}\frac{du}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{dt}{dx} - xt = x^3$$

This equation is a linear differential equation in t.

$$\frac{dt}{dx} + Pt = Q$$

Here $P = -x \qquad$ and $\qquad Q = x^3$

$\therefore$ Integrating Factor $(\text{I.F.}) = e^{\int P\,dx}$

$$= e^{-\int x\,dx} = e^{\frac{-x^2}{2}}$$

Its general solution is given by

$$t(\text{I.F.}) = \int Q(\text{I.F.})\,dx + c$$

$$t e^{-\frac{x^2}{2}} = \int x^3 e^{-\frac{x^2}{2}} dx + c$$

Putting $-\dfrac{x^2}{2} = u$

$\therefore x^2 = -2u \Rightarrow x\,dx = -du$

$$\Rightarrow t e^{-\frac{x^2}{2}} = \int 2u e^u \, du + c$$

$$= 2\,[ue^u - e^u] + c$$

$$= 2e^{-\frac{x^2}{2}}\left(-\frac{x^2}{2} - 1\right) + c$$

$$\therefore t = 2\left(-\frac{x^2}{2} - 1\right) + c e^{\frac{x^2}{2}}$$

Re-substituting the values of t and u, we get

$$\frac{1}{u} = 2\left(-\frac{x^2}{2} - 1\right) + c e^{\frac{x^2}{2}}$$

$$\therefore \frac{1}{y - x} = -x^2 - 2 + c e^{\frac{x^2}{2}}$$

$\therefore$ The general solution is $\dfrac{1}{y - x} = -x^2 - 2 + c e^{\frac{x^2}{2}}$

Example 1.72 Solve $\dfrac{dy}{dx} + x(x+y) = -1 + x^3(x+y)^2$. **[MU 2013]**

Solution **[6 Marks]**

We have $\dfrac{dy}{dx} + x(x+y) = -1 + x^3(x+y)^2$

Putting $x + y = u$

$$\therefore 1 + \frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{du}{dx} + xu = x^3u^2 \qquad \text{[Bernoulii's equation]}$$

Dividing both the sides by u^2, we get

$$\frac{1}{u^2}\frac{du}{dx} + \frac{x}{u} = x^3$$

Putting $\dfrac{1}{u} = t$

$$\therefore -\frac{1}{u^2}\frac{du}{dx} = \frac{dt}{dx}$$

$$\Rightarrow -\frac{dt}{dx} + xt = x^3$$

$$\frac{dt}{dx} - xt = -x^3$$

This equation is a linear differential equation in t.

$$\frac{dt}{dx}+Pt=Q$$

Here $P=-x$ and $Q=-x^3$

$\therefore$ Integrating Factor $(\text{I.F.})=e^{\int P\,dx}$

$$=e^{-\int x dx}=e^{\frac{-x^2}{2}}$$

Its general solution is given by

$$t(\text{I.F.})=\int Q(\text{I.F.})dx+c$$

$$\therefore\ te^{-\frac{x^2}{2}}=\int -x^3e^{-\frac{x^2}{2}}dx+c$$

Putting $-\dfrac{x^2}{2}=u$

$\therefore x^2=-2u \qquad \Rightarrow xdx=-du$

$$\therefore te^{-\frac{x^2}{2}}=\int -2ue^u du+c$$

$$=-2\left[ue^u-e^u\right]+c$$

$$=-2e^{-\frac{x^2}{2}}\left(-\frac{x^2}{2}-1\right)+c$$

$$t=2\left(\frac{x^2}{2}+1\right)+ce^{\frac{x^2}{2}}$$

Re-substituting the values of t and u, we get

$$\frac{1}{u}=2\left(\frac{x^2}{2}+1\right)+ce^{\frac{x^2}{2}}$$

$$\therefore \frac{1}{x+y}=x^2+2+ce^{\frac{x^2}{2}}$$

$\therefore$ The general solution is $\dfrac{1}{x+y}=x^2+2+ce^{\frac{x^2}{2}}$

EXERCISES

Solve the following equations:

1.36 $\dfrac{dy}{dx}(x^2y^3+xy)=1$

[**Ans:** $x(2-y^2)+\left(cxe^{-\frac{y^2}{2}}\right)=1$]

1.37 $\dfrac{dy}{dx}+y\tan x=y^2\sec x$

[**Ans:** $\dfrac{1}{y}\cos x=-x+c$]

1.38 $\dfrac{dy}{dx}+y\cos x=y^3\sin 2x$

[**Ans:** $\dfrac{1}{y^2}=(1+2\sin x)+ce^{2\sin x}$]

1.39 $\dfrac{dy}{dx}-xy=y^2e^{-\left(\frac{x^2}{2}\right)}\cdot\log x$

[**Ans:** $\dfrac{1}{y}e^{\frac{x^2}{2}}=x(1-\log x)+c$] **[MU 1998]**

1.40 $x\dfrac{dy}{dx}+y=x^3y^c$

[**Ans:** $\dfrac{1}{y^5}=\dfrac{5}{2}x^3+cx^5$] **[MU 2004, 13]**

1.41 $\left(1-x^2\right)\dfrac{dy}{dx}+xy=y^3\sin^{-1}x$

[**Ans:** $-2\left[x\sin^{-1}x+\sqrt{1-x^2}\right]+c$]

1.6 APPLICATIONS OF FIRST-ORDER AND FIRST-DEGREE DIFFERENTIAL EQUATIONS

In this section, we will apply the concept of first-order and first-degree differential equations to engineering problems.

EXAMPLES BASED ON ELECTRICAL ENGINEERING

Example 1.73 Solve $\dfrac{di}{dt}+\dfrac{R}{L}i=\dfrac{E}{L}$ for the case in which the circuit has initial current i_0 at time $t=0$ and the e.m.f. impressed is given by $E=E_0e^{-kt}$.

Solution

We have $\dfrac{di}{dt}+\dfrac{R}{L}i=\dfrac{E}{L}$

This equation is a linear differential equation in i.

$$\frac{di}{dt}+Pi=Q$$

Here $P=\dfrac{R}{L}$ and $Q=\dfrac{E}{L}$

$\therefore$ Integrating Factor $(I.F.)=e^{\int\frac{R}{L}dt}=e^{\frac{R}{L}t}$

Its general solution is given by

$$i(\text{I.F.})=\int Q(\text{I.F.})\,dt+c$$

$$\therefore ie^{\frac{R}{L}t}=\int\frac{E}{L}e^{\frac{R}{L}t}dt+c$$

$$=\frac{E_0}{L}\int e^{-kt}e^{\frac{R}{L}t}dt+c$$

$$=\frac{E_0}{L}\int e^{\left(\frac{R}{L}-k\right)t}dt+c$$

$$=\frac{E_0}{R-kL}e^{\left(\frac{R}{L}-k\right)t}+c$$

Given that at $t=0$, $i=i_0$ and $E=E_0$

$$\therefore i_0=\frac{E_0}{R-kL}+c$$

$$c=i_0-\frac{E_0}{R-kL}$$

$$i e^{\frac{R}{L}t} = \frac{E_0}{R-kL} e^{\left(\frac{R}{L}-k\right)t} + i_0 - \frac{E_0}{R-kL}$$

$$\therefore i = \frac{E_0}{R-kL} e^{-kt} + \left(i_0 - \frac{E_0}{R-kL}\right) e^{-\frac{R}{L}t}$$

Example 1.74 A resistance of 100 Ω and inductance of 0.5 H are connected in series with a battery of 20 V. Find the current at any instant if the relation between L, R, E is $L\frac{di}{dt} + Ri = E$.

Solution

We have $L\frac{di}{dt} + Ri = E$

$$\therefore \frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$$

This equation is a linear differential equation in i.

$$\frac{di}{dt} + Pi = Q$$

Here $P = \frac{R}{L}$ and $Q = \frac{E}{L}$

$\therefore$ Integrating Factor $(\text{I.F.}) = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$

Its general solution is given by

$$i(\text{I.F.}) = \int Q(\text{I.F.})\, dt + c$$

$$i \cdot e^{\frac{R}{L}t} = \int \frac{E}{L} \cdot e^{\frac{R}{L}t} dt + c$$

$$\therefore i \cdot e^{\frac{R}{L}t} = \frac{E}{L} \cdot \frac{e^{\frac{R}{L}t}}{R/L} + c$$

$$\therefore i = \frac{E}{R} + ce^{-\frac{R}{L}t}$$

For $R = 100\ \Omega$, $L = 0.5\text{H}$, $E = 20\text{V}$ and $i = 0$ when $t = 0$

We get $c = -1$

$$\therefore i = 0.2\left(1 - e^{-200t}\right)$$

Example 1.75 In a circuit containing inductance L, resistance R, and voltage E, the current I is given by $L\frac{di}{dt} + Ri = E$. Find the current I at time t if at $t = 0$, $i = 0$, and L, R, and E are constants.

Solution

We have $\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$

This equation is a linear differential equation in i.

$$\frac{di}{dt} + Pi = Q$$

Here $P = \frac{R}{L}$ and $Q = \frac{E}{L}$

$$\therefore \text{ Integrating Factor (I.F.)} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$$

Its general solution is given by

$$i(\text{I.F.}) = \int Q(\text{I.F.})\, dt + c$$

$$\therefore i \cdot e^{\frac{R}{L}t} = \int \frac{E}{L} \cdot e^{\frac{R}{L}t} dt + c$$

$$i \cdot e^{\frac{R}{L}t} = \frac{E}{L} \cdot \frac{e^{\frac{R}{L}t}}{R/L} + c$$

$$i = \frac{E}{R} + ce^{-\frac{R}{L}t}$$

When $t = 0$, $i = 0$

We have $c = -\frac{E}{R}$

$$\therefore i = \frac{E}{R} - \frac{E}{R} e^{-\frac{R}{L}t}$$

Example 1.76 The current in a circuit containing an inductance L, resistance R, and voltage $E \sin \omega t$ is given by $L\frac{di}{dt} + Ri = E \sin \omega t$. If $i = 0$ at $t = 0$, find i.

Solution

We have $L\frac{di}{dt} + Ri = E \sin \omega t$

$$\therefore \frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} \sin \omega t$$

This equation is a linear differential equation in i.

$$\frac{di}{dt} + Pi = Q$$

Here $P = \frac{R}{L}$ and $Q = \frac{E}{L} \sin \omega t$

$$\therefore \text{ Integrating Factor (I.F.)} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$$

Its general solution is given by

$$i(\text{I.F.}) = \int Q(\text{I.F.})\, dt + c$$

$$\therefore\ i\cdot e^{\frac{R}{L}t}=\int\frac{E}{L}\cdot\sin\omega t\cdot e^{\frac{R}{L}t}dt+c$$

$$=\frac{E}{L}\cdot\frac{1}{\frac{R^2}{L^2}+\omega^2}e^{\frac{R}{L}t}\left(\frac{R}{L}\sin\omega t-\omega\cos\omega t\right)+c$$

$$=E\cdot\frac{1}{R^2+L^2\omega^2}e^{\frac{R}{L}t}\left(R\sin\omega t-\omega L\cos\omega t\right)+c$$

Given that $i=0$ when $t=0$, we have $c=E\cdot\dfrac{\omega L}{R^2+L^2\omega^2}$

$$\therefore\ i\cdot e^{\frac{R}{L}t}=E\cdot\frac{1}{R^2+L^2\omega^2}e^{\frac{R}{L}t}\left(R\sin\omega t-\omega L\cos\omega t\right)+E\cdot\frac{\omega L}{R^2+L^2\omega^2}$$

Example 1.77 The charge q on the plate of a condenser of capacity C charged through a resistance R by a steady voltage V satisfies the differential equation $R\dfrac{dq}{dt}+\dfrac{q}{C}=V$. If $q=0$ at $t=0$, show that $i=\dfrac{V}{R}e^{-1/RC}\left[\because i=\dfrac{dq}{dt}\right]$.

Solution

We have $R\dfrac{dq}{dt}+\dfrac{q}{C}=V$

The given equation can be written as

$$\frac{dq}{dt}+\frac{1}{RC}q=\frac{V}{R}$$

This equation is a linear differential equation in q.

$$\frac{dq}{dt}+Pt=Q$$

Here $P=\dfrac{1}{RC}$ and $Q=\dfrac{V}{R}$

$\therefore$ Integrating Factor (I.F.) $=e^{\int Pdt}$

$$=e^{\int\frac{1}{RC}dt}=e^{\frac{1}{RC}t}$$

Its general solution is given by

$$q(\text{I.F.})=\int Q(\text{I.F.})dt+c$$

$$q\cdot e^{\frac{1}{RC}t}=\int\frac{V}{R}e^{\frac{1}{RC}t}dt+c$$

$$=\frac{V}{R}\frac{e^{\frac{1}{RC}t}}{1/RC}+c$$

$$\therefore\ q\cdot e^{\frac{1}{RC}t}=VC\,e^{\frac{1}{RC}t}+c$$

Initially $q = 0$ when $t = 0$

$$0 = VC + c$$

$$\therefore c = -VC$$

$$\therefore\ q \cdot e^{\frac{1}{RC}t} = VCe^{\frac{1}{RC}t} - VC$$

$$q = VC\left(1 - e^{-\frac{1}{RC}t}\right) \tag{1.9}$$

Differentiating Eq. (1.9) w.r.t. t, we get

$$\frac{dq}{dt} = VC\left(0 - e^{-\frac{1}{RC}t}\left(-\frac{1}{RC}\right)\right)$$

$$\frac{dq}{dt} = VC\left(\frac{e^{-\frac{1}{RC}t}}{RC}\right)$$

$$\therefore\ i = \frac{V}{R}e^{-\frac{1}{RC}t} \qquad \left[\because\ i = \frac{dq}{dt}\right]$$

Example 1.78 The equation of electromotive force in terms of current i for an electrical circuit having resistance R and a condenser of a capacity C in series is $E = Ri + \int \frac{i}{C}dt$. Find the current i at any time t, when $E = E_0 \sin\omega t$.

Solution

We have $Ri + \int \frac{i}{c}dt = E_0 \sin\omega t$

On differentiating both the sides, we get

$$R\frac{di}{dt} + \frac{i}{c} = \omega E_0 \cos\omega t$$

Dividing both the sides by R, we get

$$\frac{di}{dt} + \frac{i}{Rc} = \frac{\omega E_0}{R}\cos\omega t$$

This equation is a linear differential equation in i.

$$\frac{di}{dt} + Pi = Q$$

Here $P = \dfrac{1}{RC}$ and $Q = \omega\dfrac{E_0}{R}\cos\omega t$

$\therefore$ Integrating factor $(\text{I.F.}) = e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}}$

Its general solution is given by

$$i(\text{I.F.}) = \int Q(\text{I.F.})dt + c$$

$$i\,e^{\frac{t}{RC}} = \int \frac{\omega E_0}{R}\cos\omega t e^{\frac{t}{RC}}dt$$

$$= \frac{\omega E_0}{R} \cdot \frac{e^{\frac{t}{RC}}}{\sqrt{\left(\frac{1}{RC}\right)}} \cos\left(\omega t - \tan^{-1} \frac{\omega}{1/RC}\right) + c_1$$

$$= \frac{\omega C E_0}{\sqrt{1 + R^2C^2\omega^2}} e^{\frac{t}{Rt}} \cos(\omega t - \phi) + c_1$$

where $\tan\phi = Rc$

The required current at time t is

$$i = \frac{\omega C E_0}{\sqrt{1 + R^2C^2\omega^2}} \cos(\omega t - \phi) + c_1 e^{-\frac{t}{Rt}}$$

EXERCISES

Solve the following equations:

1.42 The equation of an L-R circuit is given by $L\frac{di}{dt} + Ri = 10\sin t$ if $i = 0$ at $t = 0$. Express i as a function of t.

[**Ans:** $i = \frac{10}{\sqrt{R^2 + L^2}}\left[\sin(t - \phi) + \sin\phi e^{\frac{Rt}{L}}\right]$]

1.43 An electric circuit contains an inductance of 6 H, resistance of 15 Ω in series with an e.m.f. of $240\cos 30t$ V. Find the current at $t = 0.01$, it is zero at $t = 0$.

[**Ans:** $i = 0.3894$]

1.44 In a circuit of resistance R and self-inductance of L, the current i is given by $L\frac{di}{dt} + Ri = E\cos\omega t$, where E and ω are constants. Find the current i at time t.

[**Ans:** $i = \frac{E}{R^2 + L^2\omega^2}(R\cos\omega t - \omega L\sin\omega t) + ce^{-\frac{R}{L}t}$]

1.45 In a circuit of resistance R and self-inductance of L, the current i is given by $L\frac{di}{dt} + Ri = 100\sin 150t$. Find the current i at the end of 0.01 s if the current is zero when $t = 0$ and $L = 2$ H, $R = 20$ Ω.

[**Ans:** 0.299 A]

1.46 When a resistance R ohms and a capacitance C farads are connected in series with an e.m.f. E volts, then the current i amperes is given by $R\frac{di}{dt} + \frac{i}{C} = \frac{dE}{dt}$. When $R = 1000$ Ω, $C = 50 \times 10^{-1}$F, $i = 10$A and $t = 0$, find the current for $t = 1$s and $E = 100\sin 120\pi t$V.

[**Ans:** 8.187 A]

EXAMPLES BASED ON MECHANICAL ENGINEERING

Example 1.79 The differential equation of a body falling from rest subjected to the force of gravity and air resistance is given by $v\frac{dv}{dx} + \frac{n^2}{g}v^2 = g$. Prove that the velocity is given by $v^2 = \frac{g^2}{n^2}\left(1 - e^{-2n^2x/g}\right)$ given that at $x = 0$, $v = 0$.

Solution

We have $v\frac{dv}{dx} + \frac{n^2}{g}v^2 = g$

Putting $v^2 = t$ $\quad\therefore 2v\frac{dv}{dx} = \frac{dt}{dx} \Rightarrow v\frac{dv}{dx} = \frac{1}{2}\frac{dt}{dx}$

$$\Rightarrow \frac{1}{2}\frac{dt}{dx}+\frac{n^2}{g}t=g$$

$$\frac{dt}{dx}+\frac{2n^2}{g}t=2g$$

This equation is a linear differential equation in t.

$$\frac{dt}{dx}+Pt=Q$$

Here $P=\dfrac{2n^2}{g}$ and $Q=2g$

$$\therefore \text{ Integrating Factor}\left(\text{I. F.}\right)=e^{\int P\,dx}=e^{\int\frac{2n^2}{g}dx}$$

$$=e^{\frac{2n^2}{g}x}$$

Its general solution is given by

$$t\left(\text{I.F.}\right)=\int Q\left(\text{I.F.}\right)dx+c$$

$$t\,e^{\frac{2n^2}{g}x}=\int 2g\cdot e^{\frac{2n^2}{g}x}dx+c$$

$$=2g\frac{e^{\frac{2n^2}{g}x}}{\frac{2n^2}{g}}+c$$

$$v^2\,e^{\frac{2n^2}{g}x}=\frac{g^2}{n^2}e^{\frac{2n^2}{g}x}+c \tag{1.10}$$

Given that at $x=0, v=0$

$$0=\frac{g^2}{n^2}+c \Rightarrow c=-\frac{g^2}{n^2}$$

From Eq. (1.10) we get

$$v^2\,e^{\frac{2n^2}{g}x}=\frac{g^2}{n^2}e^{\frac{2n^2}{g}x}-\frac{g^2}{n^2}$$

$$v^2=\frac{g^2}{n^2}-\frac{g^2}{n^2}e^{-\frac{2n^2}{g}x}$$

$$\therefore\ v^2=\frac{g^2}{n^2}\left(1-e^{-\frac{2n^2}{g}x}\right)$$

Example 1.80 A tank contains liquid of volume $V(t)$ with a concentration in percentage $C(t)$ at time t. To reduce the concentration, an inflow of rate Q_{in} is injected into the tank. The inflow has the concentration C_{in}. Assume that inflow is perfectly mixing with the liquid in the tank instantaneously. The excess liquid outflow with the rate Q_{out} is removed from tank. Suppose that at time $t=0$, the volume of liquid is V_0 with concentration C_0, then the governing equation is

$$\left[V_0+\left(Q_{\text{in}}-Q_{\text{out}}\right)t\right]\frac{dC(t)}{dt}+Q_{\text{in}}C(t)=Q_{\text{in}}\,C_{\text{in}}$$

Find concentration in percentage $C(t)$ for $Q_{\text{in}} = Q_{\text{out}} = t^2 \dfrac{\text{Lit}}{\text{sec}}$, $V_0 = 2\,\text{litres}$, $C_{\text{in}} = 2\,\text{mole}$

Solution

The governing equation is

$$\left[V_0 + \left(Q_{\text{in}} - Q_{\text{out}}\right)t\right]\frac{dC(t)}{dt} + Q_{\text{in}}C(t) = Q_{\text{in}}\,C_{\text{in}}$$

For $Q_{\text{in}} = Q_{\text{out}} = t^2 \dfrac{\text{Lit}}{\text{sec}}, V_0 = 2$ litres, $C_{\text{in}} = 2\,\text{mole}$

$$\frac{dC(t)}{dt} + t^2 C(t) = 2t^2$$

This equation is a linear differential equation in $C(t)$.

$$\frac{dC(t)}{dt} + P(t)C(t) = Q(t)$$

Here $P = t^2$ and $Q = 2t^2$

$\therefore$ Integrating Factor $(\text{I. F.}) = e^{\int P\,dt} = e^{\int t^2 dt}$

$$= e^{\frac{t^3}{3}}$$

Its general solution is given by

$$C(t)(\text{I.F.}) = \int Q(\text{I.F.})dt + c$$

$$C(t)e^{\frac{t^3}{3}} = \int 2t^2 e^{\frac{t^3}{3}}dt + c$$

$$\Rightarrow C(t)e^{\frac{t^3}{3}} = 2e^{\frac{t^3}{3}} + c$$

Hence the concentration at time t is $C(t) = 2 + ce^{-\frac{t^3}{3}}$

EXERCISES

Solve the following equations:

1.47 The differential equation of a moving body opposed by a force per unit mass of value cx and resistance per unit mass of value bv^2, where x and v are the displacement and velocity of the particle at that time, is given by $v\dfrac{dv}{dt} = -cx - bv^2$. Find the velocity of the particle in terms of x, if it starts from rest (i.e. when $x = 0, v = 0$).

[**Ans:** $v^2 = \dfrac{c}{2b^2}\left(1 - e^{-2bx}\right) - \dfrac{cx}{b}$]

1.48 A chain coiled up near the edge of a smooth table starts to fall over the edge. The velocity v when a length x has fallen is given by $xv\dfrac{dv}{dx} + v^2 = gx$. Show that $v = 8\sqrt{x/3}$ [Given $g = 32.17\text{ft/sec}$].

SUMMARY

1. A differential equation $M(x, y)dx + N(x, y)\,dy = 0$ is said to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

   The solution of the exact differential equation is given by

   $$\underset{\text{'}y\text{' constant}}{\int M\,dx} + \int \begin{bmatrix}\text{Terms in } N\text{, which}\\ \text{are free from } x\end{bmatrix} dy = c$$

   OR

   $$\int \begin{bmatrix}\text{Terms in } M\text{, which}\\ \text{are free from } y\end{bmatrix} dx + \underset{\text{'}x\text{' constant}}{\int N dy} = c$$

2. If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then $M(x, y)dx + N(x, y)\,dy = 0$ is **NOT** exact. We can convert non-exact differential equations into exact differential equations by multiplying them with a suitable factor known as integrating factor (I.F.). The following methods enable us to identify integrating factor

   Rule I: If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, then I.F. $= e^{\int f(x)dx}$

   Rule II: If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$, then I.F. $= e^{\int f(y)dy}$

   Rule III: If $y\,f_1(x,y)dx + x\,f_2(x,y)dy = 0$ and $Mx - Ny \neq 0$ then I.F. $= \dfrac{1}{Mx - Ny}$

   Rule IV: If $M(x, y)dx + N(x, y)\,dy = 0$ is homogeneous and $Mx + Ny \neq 0$ then I.F. $= \dfrac{1}{Mx + Ny}$

   Then solve the resulting differential equation using the method mentioned in Rule 1.

3. A differential equation is called linear if the dependent variable and its derivative appear in the first degree.
   The first-order linear differential equation is given by $\frac{dy}{dx} + P\,y = Q$. This equation which is linear in y can be solved by multiplying with I.F. $= e^{\int P(x)dx}$. Thus, solution to linear differential equation is given by

   $$y(\text{I.F.}) = \int Q(\text{I.F.})dx + c$$

   OR

   $$y \cdot e^{\int P\,dx} = \int e^{\int P\,dx} \cdot Q\,dx + c$$

   Similarly an equation which is linear in x is given by $\frac{dx}{dy} + P'x = Q'$ and its solution is given by

   $$x(\text{I.F.}) = \int Q'(\text{I.F.})dy + c$$

   OR

   $$x \cdot e^{\int P\,dy} = \int e^{\int P\,dy} \cdot Q' dy + c \quad \text{where I.F.} = e^{\int P\,dy}$$

4. An equation of the type $f'(y)\frac{dy}{dx} + Pf(y) = Q$ can be reduced to linear form by substitution with $f(y) = u$, $f'(y)\frac{dy}{dx} = \frac{du}{dx}$. Thus,

   $f'(y)\frac{dy}{dx} + Pf(y) = Q \Rightarrow \frac{du}{dx} + P\,u = Q$ which is linear in u.
   The solution to such linear equations is given by

   $$u(\text{I.F.}) = \int Q(\text{I.F.})dx + c$$

   OR

   $$u \cdot e^{\int P\,dx} = \int e^{\int P\,dx} \cdot Q\,dx + c$$

   Similarly an equation of the type

   $f'(x)\frac{dx}{dy} + Pf(x) = Q$ can be reduced to linear form by substitution with $f(x) = u$, $f'(x)\frac{dx}{dy} = \frac{du}{dy}$.

   Thus,

   $f'(x)\frac{dx}{dy} + P'f(x) = Q' \Rightarrow \frac{du}{dy} + Pu = Q$ which is linear in u.
   The solution to such linear equations is given by

   $$u(\text{I.F.}) = \int Q'(\text{I.F.})dy + c$$

   OR

   $$u \cdot e^{\int P'\,dy} = \int e^{\int P'\,dy} \cdot Q'\,dy + c$$

5. Bernoulli's equation

A differential equation of the form $\frac{dy}{dx} + P\,y = Q\,y^n$ is called Bernoulli's equation. Here P and Q are functions of x alone or constants and n is a real number.

The above equation can be made linear differential equation by dividing both sides by y^n.

$$\therefore y^{-n}\frac{dy}{dx} + P\,y^{1-n} = Q \tag{1}$$

Putting $y^{1-n} = u$

$$\therefore (1-n)\,y^{-n}\frac{dy}{dx} = \frac{du}{dx}$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}$$

$$\frac{1}{1-n}\frac{du}{dx} + P\,u = Q$$

$$\therefore \frac{du}{dx} + (1-n)\,Pu = (1-n)\,Q$$

It is a linear differential equation in u. Its general solution is given by

$$u(\text{I.F.}) = \int Q(\text{I.F.})\ dx + c$$

Similarly $\frac{dx}{dy} + P'x = Q'x^n$ is also called Bernoulli's equation. Here P' and Q' are the functions of y alone or constants.

The above equation can be made linear by dividing throughout by x^n.

$$x^{-n}\frac{dx}{dy} + P'x^{1-n} = Q'$$

Put $x^{1-n} = u$ and proceed.